The Distribution of Values of Artin L-Functions and Applications

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CHAPTER 1

Introduction

Artin L-series were introduced by Artin in his articles "Über eine neue Art von L-Reihen" (1923) and "Zur Theorie der L-Reihen mit allgemeinen Gruppencharakteren" (1930) [**3**]. In the proof of his "reciprocity law" Artin showed that in the case of an Abelian extension of number fields Artin L-functions are just Hecke L-functions. Therefore the theory of those functions did directly apply to Abelian Artin L-series. For example we know that Hecke L-functions with non-trivial character are entire functions. Artin's conjecture states the same for general Artin L-functions with non-trivial character. Since Brauer [**5**] it is known, that these functions have a meromorphic continuation to $\mathbb{C}$ and a functional equation like Hecke L-functions. However it is unknown if they have poles in the critical strip $0 < Re(s) < 1$. Artin's conjecture on the holomorphy of the Artin L-functions has inspired a lot of development in number theory [**21**], namely for example the Langlands' program to find a general reciprocity law for non-Abelian extensions of number fields. For the analogue of those Artin L-functions in the case of function fields Artin's conjecture is known to be true and this fact played a prominent role in Laurent Lafforgue's proof of the Langlands correspondence for function fields [**17**].
We report on the fundamentals of Artin L-series in the next chapter.

The study of the distribution of non-zero values of Riemann's Zeta-function starts with Harald Bohr's [**4**] work. He investigated the value distribution of the Zeta-function for $Re(s) > 1$. An exposition on this subject can be found in Titchmarsh's "The Theory of the Riemann Zeta-Function" [**30**]. The work of Voronin [**14**] extends this investigation to the investigation of the distribution of non-zero values in the strip $1/2 < Re(s) < 1$. He gets a quite new type of theorems, which are called "Universality" theorems in the literature. Generalizations of these theorems to other Dirichlet series exist, for example to Dedekind Zeta-functions [**26**] and to the Lerch Zeta-function [**10**]. Further generalizations are concerned with the joint distribution of non-zero values of Dirichlet L-functions ([**14**], [**11**]) and with the joint distribution of non-zero values of Lerch Zeta-functions [**18**].

Our approach generalizes the theorem on the joint distribution of non-zero values of Dirichlet L-series [**14**] to Artin L-series of an arbitrary normal extension $K/\mathbb{Q}$. It is unconditional, i.e. we do not presuppose Artin's conjecture to be true. It is more than a theorem on the joint distribution of non-zero values, since it states that we may approach jointly n arbitrary non-zero holomorphic functions by n Artin L-functions (Theorem 5.1).

To prove this result we need a mean value theorem. This theorem (Theorem 4.1) does not apply to Artin L-functions, since we do not know if they are holomorphic in the critical strip. However it is valid for Hecke L-functions and Dedekind Zeta-functions (Remark 4.1), because they only possess a limited number of poles. The method we use for this purpose is known as Carlson's method [**30**], and was applied to the k-th power moment of the Riemann Zeta-function.

A theorem of Davenport and Heilbronn [**8**] states that Hurwitz Zeta-functions and Zeta-functions attached to positive definite quadratic forms of discriminant d, such that the class number $h(d)$ is greater than 1, have zeros with $Re(s) > 1$. It was proved by Voronin [**14**], that those functions do have zeros in the strip $1/2 < Re(s) < 1$. We prove (Theorem 6.4) that this is true for every partial zeta-function attached to a class of a ray class group of any algebraic number field, provided that this group has cardinality greater than 1. This especially applies to the class group of a number field with class number greater than 1. The zeta-functions of every class of the class group of a number field have a functional equation like the Riemann Zeta-function [**16**, p.254], and therefore we have found other functions for which "the analogue of the Riemann hypothesis is false" [**30**, p.282]. If we take the generalized Riemann hypothesis for granted, then the sum of all these partial zeta-functions should have no zeros in the strip $1/2 < Re(s) < 1$, although each of its summands has infinitely many zeros. Thus these zeros must be at different places. We recall, that these zeta-functions play a prominent role in class field theory (Hasse [**13**], Stark [**28**]).

It is known, that the Dedekind Zeta-functions of different normal extensions differ. We show to which extend Zeta-functions of different normal extensions are really different (Theorem 6.6). The theorems in the last chapter are applications of Theorem 5.1 on Artin L-functions.

1. Notations

We use the big O-notation (Landau symbol) in the following way: By $f(t) = O(g(t))$ we mean that f is a function with the property $|f(t)| \leq Cg(t)$ for all t. The constant C depends only on f and g. By $f(t) = O_a(g(t))$ we emphasize, that $C > 0$ depends on a. The notation $vol(M)$ for some set $M \subset \mathbb{R}^n$ denotes the Lesbegue-measure of this set, which has volume 1 on the unit cube. $\Gamma(s)$ is the Gamma function [**2**]. The Greek letters Γ and γ are also used for curves in the complex plane or in $\mathbb{R}^n$. $Re(z)$ and $Im(z)$ are the real part and the complex part of $z \in \mathbb{C}$.

If $\alpha \in \mathbb{R}$ is a real number, then $\{\alpha\} := \alpha - [\alpha]$. $[\alpha]$ denotes the largest integer $n \in \mathbb{Z}$ with $n \leq \alpha$. $\gcd(a, b)$ is the greatest common divisor of integers in $\mathbb{Z}$ or of ideals, if defined. For a finite set M we denote by $\#M$ its cardinality. Algebraic number fields [**20**] are denoted by small or big Latin letters k, K, L. The Galois group of a normal extension K/k is denoted by $G(K/k)$. For a finite algebraic extension K/k we denote by $[K : k]$ its relative degree. The trace of an algebraic number α is denoted by $Trace(\alpha)$, its norm by $N(\alpha)$ or $N_{K/k}(\alpha)$, if relative to the subfield k. $\mathcal{O}_k$ denotes the ring of integers of the number field k. Ideals are denoted by $\mathfrak{a}$ or $\mathfrak{b}$. Those letters may also denote the modulus of a class group in the sense of class field theory [**13**]. The norm of an ideal $\mathfrak{a}$ is denoted by $N(\mathfrak{a})$. $\mathbb{P}$ is the set of all rational primes. $\mathbb{P}_k$ is the set of prime ideals of $\mathcal{O}_k$. The exponent k of the exact power p^k dividing a rational integer d, i.e. $\gcd(p^{k+1}, d) = p^k$, will be denoted by $v_p(d)$, i.e. $v_p(d) := k$. 1 denotes also the neutral element of a group. It may as well be used for the identity element of a Galois group G and for the character $\chi : G \to \mathbb{C}$ with $\chi(g) = 1 \in \mathbb{C}$ for all $g \in G$. The group of characters of an Abelian group G is denoted by G^*.

$GL_k(\mathbb{C})$ is the group of all $k \times k$-matrices, which have an inverse. For a matrix A we denote by $Tr(A)$ or $Trace(A)$ its trace and by $\det(A)$ its determinant. The restriction of a map $f : M \to T$ to a subset $U \subset M$ will be denoted by $f_{|U}$, i.e. $f_{|U} : U \to T$.

CHAPTER 2

Fundamentals

1. Linear Representation of Finite Groups and Artin L-Series

By a *class function* on a finite group G we mean a function $f : G \mapsto \mathbb{C}$ such that $f(\tau g \tau^{-1}) = f(g)$ for all $\tau, g \in G$. In other words: The value of a class function depends only on the conjugacy classes of the group.

DEFINITION 1. *Let G be a finite group, $\rho : G \mapsto GL_k(\mathbb{C})$ a group homomorphism. ρ is called a representation of G. Then $\chi : G \longmapsto \mathbb{C}$ with $\chi(g) := Trace(\rho(g))$ is called a character of G. The degree of this character is k.*

Obviously every character is a class function and the degree of a character is equal to $\chi(1)$.
We call this kind of characters also a *non-Abelian character* if we want to distinguish them from the usual *Abelian characters* of Abelian groups.

DEFINITION 2. *An irreducible representation of the group G is a group homomorphism $\rho : G \longrightarrow GL_k(\mathbb{C})$ that can not be decomposed into the direct sum of two representations. An irreducible character is the character of an irreducible representation.*

THEOREM 2.1. [**27**, p.18] *The irreducible characters of a finite group G form an orthonormal basis of the vector space of class functions on G with respect to the scalar product $\big(\chi, \psi\big) := \frac{1}{\#G} \sum_{g \in G} \chi(g)\overline{\psi(g)}$. The dimension of the vector space of class functions is equal to the number of conjugacy classes of G.*

Every character on a group G is the sum of (not necessarily different) irreducible characters of this group.

Definition 3 (induced character).
Let U be a subgroup of the finite group G and χ a character of U. Then for every $g \in G$ we have the induced character of χ defined by

$$\chi^*(g) := \frac{1}{\#U} \sum_{v \in G} \chi(vgv^{-1})$$

where $\chi(a) := 0$ if $a \notin U$.

An induced character is a character in the sense of the above definition of characters.

Theorem 2.2 (Frobenius reciprocity). [**27**, p.86] *Let U be a subgroup of G. If ψ is a class function on U and ϕ a class function on G, we have (with the scalar product above)*

$$\left(\psi, \phi_{|U}\right)_U = \left(\psi^*, \phi\right)_G.$$

Theorem 2.3 (Brauer). [**23**, p.544] *Every character on a finite group G is a finite linear combination $\chi = \sum_l n_l \varphi_l^* - \sum_l m_l \psi_l^*$, where φ_l^* and ψ_l^* are induced from characters φ_l, ψ_l of degree 1 of subgroups of G and $n_l, m_l \in \mathbb{Z}^{\geq 0}$.*

Let K be a normal extension of k with Galois group $G(K/k)$. Denote by $I_{\mathfrak{P}}$ the inertia group and by $D_{\mathfrak{P}}$ the decomposition group of the Galois group $G(K/k)$ corresponding to the prime ideal $\mathfrak{P}$ with $\mathfrak{p} \subset \mathfrak{P}$, prime ideal $\mathfrak{p} \subset \mathcal{O}_k$ and $\mathfrak{P} \mid \mathfrak{p}\mathcal{O}_K$ ([**12**, p.33] and [**20**, p.98]).

If $I_{\mathfrak{P}} = \{1\}$, the Frobenius-Automorphism $\sigma := (\mathfrak{P}, K/k) \in D_{\mathfrak{P}}$ is defined by [**20**, p.108]

$$\sigma\alpha \equiv \alpha^{N(\mathfrak{p})} \mod \mathfrak{P}$$

for any $\alpha \in \mathcal{O}_K$ and $\mathfrak{p} \subset \mathfrak{P}$. If we exchange the prime ideal $\mathfrak{P}$ by the prime ideal $\mathfrak{P}'$ with $\mathfrak{p} \subset \mathfrak{P}'$, then the corresponding Frobenius-Automorphisms $(\mathfrak{P}, K/k)$ and $(\mathfrak{P}', K/k)$ are conjugate.

Let ρ be any finite linear representation of $G(K/k)$. Denote the character of ρ by χ. Set $L_{\mathfrak{p}}(s, \chi, K/k) := \det\left(E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k))\right)^{-1}$, where E is the unit matrix. Obviously this definition is independent of the choice of the prime ideal $\mathfrak{P}$ with $\mathfrak{p} \subset \mathfrak{P}$. Also it is clear that this definition depends only on χ and not on the specific representation ρ with $\chi(\sigma) = Tr(\rho(\sigma))$, since the value of a determinant is invariant under conjugation.

If $I_{\mathfrak{P}} \neq \{1\}$, then set $V^{I_{\mathfrak{P}}} := \{x \in \mathbb{C}^k \mid \forall_{\tau \in I_{\mathfrak{P}}} : \rho(\tau)(x) = x\}$. Then replace $E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k))$ in the definition of $L_{\mathfrak{p}}(s, \chi, K/k)$ by the restriction $E - N(\mathfrak{p})^{-s}\rho((\mathfrak{P}, K/k)) \mid_{V^{I_{\mathfrak{P}}}}$ to the subspace $V^{I_{\mathfrak{P}}}$.

We may write $\sigma_{\mathfrak{p}}$ instead of $(\mathfrak{P}, K/k)$, since class functions and $L_{\mathfrak{p}}(s, \chi, K/k)$ only depend on the conjugacy classes of a given group element. We write $L_p(s, \chi)$ for $L_p(s, \chi, K/\mathbb{Q})$.

DEFINITION 4 (Artin L-Series). [**23**, p.540] *The Artin L-series of a character χ on the group $G(K/k)$ is defined by*

$$L(s, \chi, K/k) := \prod_{\mathfrak{p} \in \mathbb{P}_k} L_{\mathfrak{p}}(s, \chi, K/k) \text{ for all } s \in \mathbb{C} \text{ with } Re(s) > 1.$$

The function $L(s, \chi, K/k)$ has a meromorphic continuation to $\mathbb{C}$.

In [**3**, p.169] Artin defines the Artin L-series by its logarithm:

$$\log L(s, \chi, K/k) = \sum_{\mathfrak{p}^h} \frac{\chi(\mathfrak{p}^h)}{hN(\mathfrak{p})^{hs}} \text{ for } Re(s) > 1.$$

We do not describe the details of this definition. However we remark that the Dirichlet-coefficients $\frac{\chi(\mathfrak{p})}{h}$ of this Dirichlet series $\log L(s, \chi, K/k)$ are dominated by the Dirichlet-coefficients of $\chi(1) \log L(s, 1, K/k)$. According to the next theorem $L(s, 1, K/k)$ is identical with the Dedekind Zeta-function.

We write $L(s, \chi)$ for $L(s, \chi, K/\mathbb{Q})$. An Artin L-series $L(s, \chi, K/k)$ is called *primitive* if χ is an irreducible character of the Galois group of K/k.

Artin's conjecture says, that $L(s, \chi, K/k)$ is an entire function for all irreducible characters $\chi \neq 1$ [**23**, p.547]. However it is unproven until now. Therefore we do not know if Artin L-Series are entire or if they have poles in the critical strip $0 < Re(s) < 1$.

THEOREM 2.4. [**23**, p.544]

1. *$L(s, 1, K/k) = \zeta_k(s)$.*
2. *If $k \subset K \subset L$ are Galois extensions of k, and χ is a character of $G(K/k)$, which may be viewed as a character of $G(L/k)$ by applying the restriction map, then*
   $L(s, \chi, K/k) = L(s, \chi, L/k)$.
3. *Let L/k be a Galois extension and K any subfield with*
   $k \subset K \subset L$. Then for a character χ of $G(L/K)$ we have
   $L(s, \chi, L/K) = L(s, \chi^*, L/k)$.
4. *$L(s, \chi + \psi, K/k) = L(s, \chi, K/k)L(s, \psi, K/k)$.*

REMARK 2.1. *All the proofs in the last Theorem are done for the Euler-factors $L_{\mathfrak{p}}(s, \chi, K/k)$. So these statements hold "locally":*

1. *$L_{\mathfrak{p}}(s, 1, K/k) = \left(1 - N(\mathfrak{p})^{-s}\right)^{-1}$.*
2. *$L_{\mathfrak{p}}(s, \chi, K/k) = L_{\mathfrak{p}}(s, \chi, L/k)$.*
3. *$\prod_{\mathfrak{p} | \mathfrak{q}\mathcal{O}_K} L_{\mathfrak{p}}(s, \chi, L/K) = L_{\mathfrak{q}}(s, \chi^*, L/k)$.*
4. *$L_{\mathfrak{p}}(s, \chi + \psi, K/k) = L_{\mathfrak{p}}(s, \chi, K/k)L_{\mathfrak{p}}(s, \psi, K/k)$.*

COROLLARY 2.1. [**23**, p.547]
If $k \subset K$ is a finite Galois extension with Galois group $G := G(K/k)$, then

$$\zeta_K(s) = \zeta_k(s) \prod_{\chi \neq 1} L(s, \chi, K/k)^{\chi(1)}.$$

Denote the conjugacy classes of the Galois group $G(K/\mathbb{Q})$ by $C_1, \ldots, C_N$. Then

THEOREM 2.5 (Artin). [**3**, p.122] *Denote by $\pi(C_j, x)$ the number of rational primes $p \leq x$ with $\sigma_p \in C_j$. Then*

$$\pi(x, C_j) = \frac{h_j}{k} \int_2^x \frac{dt}{\log t} + O(xe^{-a \log^{1/2} x})$$

where a is some positive constant, $k := \#G$ and $h_j := \#C_j$.

For a more effective and also unconditional version, see the article of Lagarias and Odlyzko [**15**].

2. Theorems from Complex Analysis and Hilbert Space Theory

A series $a_n, n \in \mathbb{N}$ of real numbers is called *conditionally convergent*, if $\sum_{n\in\mathbb{N}} |a_n|$ is unbounded and $\sum_{n\in\mathbb{N}} a_n$ converges for an appropriate rearrangement of the terms a_n. The following Theorem generalizes Riemann's Rearrangement Theorem, which states that a series of real numbers is conditionally convergent if and only if it can be rearranged such that its sum converges to an arbitrary preassigned real number.

THEOREM 2.6. [**14**, p.352] *Suppose that a series of vectors* $\sum_{n=1}^{\infty} u_n$ *in a real Hilbert space* $\mathcal{H}$ *satisfies* $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty$ *and for every* $e \in \mathcal{H}$ *with* $e \neq 0$ *the series* $\sum_{n=1}^{\infty} \langle u_n, e\rangle$ *converges conditionally. Then for any* $v \in \mathcal{H}$ *there is a permutation* π *of* $\mathbb{N}$ *such that* $\sum_{n=1}^{\infty} u_{\pi(n)} = v$ *in the norm of* $\mathcal{H}$.

THEOREM 2.7 (Paley-Wiener). [**24**, p.13] *Let* F *be an entire function. Then the following statements are equivalent:*

(1) $\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty$ *and* $\limsup_{z\in\mathbb{C}} |F(z)e^{-(\sigma+\epsilon)|z|}| < \infty$ *for every* $\epsilon > 0$

(2) *there is a function* $f \in L^2(-\sigma,\sigma)$ *such that*
$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} f(u)e^{iuz} du$$

This theorem has the following consequence.

COROLLARY 2.2. *Suppose that an entire function* $g \not\equiv 0$ *has a series expansion* $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ *and the sequence* $\{|a_n|\}_{n\in\mathbb{N}}$ *is bounded. Then for every* $c > 1$ *there is an unbounded sequence* $\{u_k\}_{k\in\mathbb{N}}$ *of positive real numbers such that* $|g(u_k)| > \exp(-cu_k)$.

In other words: The function $g(z)$ is not only bounded by $\exp(cz)$ from above but also in a certain sense from below.

Proof: Suppose the converse. Then we have $|g(u)| < A\exp(-cu)$ for some $A > 0$ and all positive real u. Then

$|g(u)\exp((1+\delta)u)| < A\exp(-\delta u)$ for $\delta := (c-1)/2$. Due to the conditions on the coefficients a_n we have $|g(-u)| < B\exp(u)$ for some $B > 0$ and positive real u. Therefore again $|g(-u)\exp(-(1+\delta)u)| < B\exp(-\delta u)$ and for the maximum C of A and B it follows $|g(u)\exp((1+\delta)u)| < C\exp(-\delta|u|)$ for all $u \in \mathbb{R}$.

Set $F(z) := g(z)\exp((1+\delta)z)$. Then $\limsup_{z\in\mathbb{C}} |F(z)e^{-(2+\delta+\epsilon)|z|}| < \infty$ for every $\epsilon > 0$. We have $|F(u)|^2 < C^2\exp(-2\delta|u|)$. Therefore condition (1) of the preceeding Theorem is satisfied. Then we have a function $f \in L^2(-(\delta+2), \delta+2)$ such that $F(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)e^{iuz}du$. According to Plancherel's Theorem [**24**, p.2] we find that $f(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(u)e^{-iux}du$ almost everywhere in $\mathbb{R}$. Since $|F(u)| < C\exp(-\delta|u|)$, the function defined by the integral is analytic in a strip near the real line. However the support of $f(x)$ lies inside a compact interval. Therefore the analytic function defined by $\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} F(u)e^{-iux}du$ is zero outside of this interval for real x. Therefore it must be zero everywhere, which is a contradiction to $g \not\equiv 0$. □

Theorem 2.8 (Markov). [**1**, p.314] *Let P be a polynomial of degree $\leq n$ with real coefficients. Then* $\max_{|x|\leq 1}|P'(x)| \leq n^2 \max_{|x|\leq 1}|P(x)|$.

Theorem 2.9. [**30**, p.303] *Suppose that $f(z)$ is holomorphic on $|z-z_0| \leq R$. Then for $|z-z_0| \leq R' < R$*

$$|f(z)|^2 \leq \frac{\int\limits_{|z-z_0|\leq R} |f(x+iy)|^2 dxdy}{\pi(R-R')^2} .$$

As an obvious consequence we get:

Corollary 2.3. *Suppose that $f_1, \ldots, f_N$ are functions continuous on $|z-z_0| \leq R$ and holomorphic for all z with $|z-z_0| < R$. Suppose that for a sequence of holomorphic functions $\{\phi_{l,n}\}_{n\in\mathbb{N}}$ for $1 \leq l \leq N$*

$$\lim_{n\to\infty} \int\limits_{|z-z_0|\leq R} \sum_{l=1}^{N} |\phi_{l,n}(z) - f_l(z)|^2 dxdy = 0.$$

Then for every $\epsilon > 0$ there is a number $n_0 \in \mathbb{N}$ such that for a fixed $R' < R$ and all $n \geq n_0, |z-z_0| \leq R'$ and all $1 \leq l \leq N$

$$|f_l(z) - \phi_{l,n}(z)| < \epsilon.$$

Definition 5 (Hardy-space). *The vector space $\mathcal{H}_2$ of functions $f(s)$, which are analytic on the disc $|s| < R$ and with*

$$\lim_{r\to R} \int\limits_{|z|\leq r} |f(z)|^2 dxdy \ < \infty$$

is a real Hilbert space with norm

$$\|f\|_2 := \big(\lim_{r\to R} \int\limits_{|z|\leq r} |f(z)|^2 dxdy\big)^{1/2}$$

and scalar product

$$\langle f, g\rangle := \lim_{r\to R} Re \int\limits_{|z|\leq r} f(z)\overline{g(z)}dxdy.$$

The general theory of such Hilbert spaces is developed in [**9**, p.257].

It is well known, that every function f analytic on $|s| < R$ has an convergent Taylor series $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$. This series is absolutely convergent and $\limsup\limits_{n\geq 0} |a_n|^{1/n} \leq 1/R$. Likewise for $|z| = r < R$ we have $\sum\limits_{n,m=0}^{\infty} |a_n \overline{b}_m z^n \overline{z}^m| = \sum\limits_{n,m=0}^{\infty} |a_n||b_m| r^{n+m} = \sum\limits_{n=0}^{\infty} |a_n| r^n \sum\limits_{m=0}^{\infty} |b_m| r^m < \infty$ for every two functions analytic on $|z| < R$. Therefore

$$\int\limits_{|z|\leq r} f(z)\overline{g(z)}dxdy = \sum_{n,m=0}^{\infty} a_n\overline{b}_m \int\limits_{|z|\leq r} z^n\overline{z}^m dxdy$$

$$= \sum_{n,m=0}^{\infty} a_n\overline{b}_m \int\limits_0^{2\pi}\int\limits_0^{r} \rho^{n+m+1} e^{i(n-m)\varphi} d\rho d\varphi = \pi \sum_{n=0}^{\infty} a_n\overline{b}_n \frac{r^{2(n+1)}}{(n+1)}.$$

Therefore our space consists just of those functions with

$$\sum_{n=0}^{\infty} |a_n|^2 \frac{R^{2n}}{(n+1)} < \infty$$

and has the scalar product

$$\langle f, g\rangle = \pi R^2 \sum_{n=0}^{\infty} Re(a_n\overline{b}_n) \frac{R^{2n}}{(n+1)}.$$

Theorem 2.10 (Rouché). [**2**] *Let the curve γ be homologous to zero in a domain Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point $z \in \Omega$ not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.*

We have $n(\gamma, z) := \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{\zeta - z} d\zeta$.

Theorem 2.11. [**29**, p.304.(9.51)] *Suppose that $f(s)$ is regular and for some $A > 0$ and all $\sigma := Re(s) \geq \alpha$ we have $|f(s)| = O\big(|Im(s)|^A\big)$, whereas $\alpha \in \mathbb{R}$ is fixed. Suppose that for $\sigma > \sigma_0$ with some $\sigma_0 \in \mathbb{R}$*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n^\sigma} < \infty \text{ and } f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

If for $\sigma > \alpha$

$$\frac{1}{2T} \int\limits_{-T}^{T} |f(\sigma + it)|^2 dt$$

is bounded for $T \longrightarrow \infty$, then for $\sigma > \alpha$

$$\lim_{T \longrightarrow \infty} \frac{1}{2T} \int\limits_{-T}^{T} |f(\sigma + it)|^2 dt = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}$$

uniformly in every strip $\alpha < \sigma_1 \leq \sigma \leq \sigma_2$.

Lemma 2.1. [**30**, p.151] *Let $f(s) = \sum\limits_{n=1}^{\infty} a_n n^{-s}$ be absolutely convergent for $Re(s) > 1$. Then*

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{\delta n} = \frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(w - s) f(w) \delta^{s-w} dw$$

for $\delta > 0, c > 1, c > Re(s)$.

Lemma 2.2. [**30**, p.140] *$\delta > 0$ and $1/2 < \sigma < 1$. Then*

$$\sum_{0<m<n<\infty} \frac{e^{-(m+n)\delta}}{m^\sigma n^\sigma log(n/m)} = O\big(\delta^{2\sigma-2} \log \frac{1}{\delta}\big).$$

Theorem 2.12 (Phragmen-Lindelöf). [**16**, p.262] *Let $f(s)$ be holomorphic in the upper part of the strip: $a \leq \sigma \leq b$, and $t \geq t_1 > 0$. Assume that $f(s)$ is $O(e^{t^\alpha})$ with $1 \leq \alpha$, and $t \to \infty$ in this strip, and $f(s)$ is $O(t^M)$ for some real number $M \geq 0$, on the sides of the strip, namely $\sigma = a$ and $\sigma = b$. Then $f(s)$ is $O(t^M)$ in the strip. In particular, if f is bounded on the sides, then f is bounded on the strip.*

We state a consequence of Cauchy's integral formula.

Theorem 2.13. [**2**, p.122] *For any analytic function we have*

$$|f^{(n)}(0)| \leq \frac{n!}{r^n} \max_{|z|=r} |f(z)|$$

if f is continuous on $|z| \leq r$ and analytic in the disc $|z| < r$.

3. Theorems from Number Theory

Let $x \in \mathbb{R}^N, \gamma \subset \mathbb{R}^N$. The notation $x \in \gamma \mod \mathbb{Z}$ means that there is a vector $y \in \mathbb{Z}^N$ such that $x - y \in \gamma$. Fix a real number $\theta_0 \in \mathbb{R}$ and $\epsilon > 0$. We use the notation $|\theta_0 - \theta \mod \mathbb{Z}| < \epsilon$ to denote those numbers $\theta \in \mathbb{R}$ which have a representative number $\theta' \in \mathbb{R}$ such that $|\theta_0 - \theta'| < \epsilon$ and $\theta - \theta' \in \mathbb{Z}$.

Theorem 2.14. [**30**, p.301],[**14**] *Let $\alpha_1, \ldots, \alpha_N$ be real numbers which are $\mathbb{Q}$-linear independent, and let γ be a subregion of the unit cube of $\mathbb{R}^N$ with Jordan volume Γ. Denote by $I_\gamma(T)$ the measure of the set $\{t \mid t \in (0,T)$ and $(\alpha_1 t, \ldots, \alpha_N t) \in \gamma \mod \mathbb{Z}\}$. Then*

$$\lim_{T \longrightarrow \infty} \frac{I_\gamma(T)}{T} = \Gamma$$

A curve $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^N$ is said to be *uniformly distributed* $\mod \mathbb{Z}$ if for every parallelepiped $\Pi = \prod_{j=1}^{N} [a_j, b_j]$ with $a_j, b_j \in [0,1]$ for $1 \leq j \leq N$

$$\lim_{T \to \infty} \frac{vol\{t \mid t \in (0,T), \gamma(t) \in \Pi \mod \mathbb{Z}\}}{T} = \prod_{j=1}^{N} (b_j - a_j)$$

According to the preceeding Theorem 2.14 the curve $\gamma(t) := (\alpha_1 t, \ldots, \alpha_N t)$ is uniformly distributed.

THEOREM 2.15. [**14**, p.362] *Suppose that the curve* $\gamma(t) = \big(\gamma_1(t), \ldots, \gamma_N(t)\big)$ *is uniformly distributed* $\mod \mathbb{Z}$ *and continuous as a function* $\mathbb{R}^{>0} \to \mathbb{R}^N$. *Let the function* F *be Riemann integrable on the unit cube in* $\mathbb{R}^N$.

Then

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\{\gamma_1(t)\}, \ldots, \{\gamma_N(t)\})dt = \int_0^1 \cdots \int_0^1 F(x_1, \ldots, x_N)dx_1 \cdots dx_N.$$

THEOREM 2.16. [**14**, p.362] *Suppose that* D *is a Jordan measurable and closed subregion of the unite cube in* $\mathbb{R}^N$. γ *is a continuous and uniformly distributed* $\mod \mathbb{Z}$ *curve.* Ω *is a family of complex-valued functions, which are uniformly bounded and equicontinuous on* D.

Then the following relation holds uniformly with respect to $F \in \Omega$*:*

$$\lim_{T\to\infty} \frac{1}{T} \int_{(0,T)\cap A_D} F(\{\gamma_1(t)\}, \ldots, \{\gamma_N(t)\})dt = \int_D F(x_1, \ldots, x_N)dx_1 \cdots dx_N$$

where $A_D := \{t \mid \gamma(t) \in D \mod \mathbb{Z}\}$.

CHAPTER 3

Fundamental Lemmata

Denote by $\mathbb{P}$ the set of rational primes.

Definition 6. *Suppose that*

$$F(s) = \prod_{p \in \mathbb{P}} f_p(p^{-s})$$

where $f_p(z)$ is a rational function and the product converges absolutely for $Re(s) > 1$.

Then for any finite set $M \subset \mathbb{P}$ of primes and for any $\theta \in \mathbb{R}^{\mathbb{P}}$ we define

$$F_M(s, \theta) := \prod_{p \in M} f_p(p^{-s} e^{-2\pi i \theta_p}).$$

This definition applies to Artin L-Series defined over $\mathbb{Q}$.
According to Definition 4 we have $L(s, \chi, K/\mathbb{Q}) = \prod_{p \in \mathbb{P}} L_p(s, \chi)$ for $Re(s) > 1$ with $L_p(s, \chi) = \det(E - \rho(\sigma_p) p^{-s} \mid_{V^{I_{\mathfrak{P}}}})^{-1}$. Then

$$f_p(z) = \det(E - \rho(\sigma_p) z \mid_{V^{I_{\mathfrak{P}}}}))^{-1}.$$

It is independent of the specific representation ρ of the character χ. Thus $L_M(s, \chi, \theta)$ is well defined for every Artin L-Series $L(s, \chi, K/\mathbb{Q})$ defined over $\mathbb{Q}$.

In the case of Hecke L-series $L(s, \chi)$ we have

$$f_p(p^{-s}) := \prod_{p \in \mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1}$$

for the prime ideals $\mathfrak{p}$ lying above $p \in \mathbb{P}$. This is obviously a rational function in the argument p^{-s} since $N(\mathfrak{p}) = p^{f(\mathfrak{p}/p)}$ with $f(\mathfrak{p}/p) \in \mathbb{Z}^{\geq 1}$.

LEMMA 3.1. *Suppose that* $F_1(s), \dots, F_n(s)$ *are analytic functions which are represented by absolutely convergent products*

$$F_l(s) = \prod_{p \in \mathbb{P}} f_{p,l}(p^{-s})$$

for $Re(s) > 1$,*where* $f_{p,l}(z) = 1 + \sum_{m=1}^{\infty} a_{p,l}^{(m)} z^m$ *are rational functions of* z *without poles in the disc* $|z| < 1$. *Set* $a_{d,l} := \prod_{p \in \mathbb{P}} a_{p,l}^{(v_p(d))}$. *For all* $\epsilon > 0$ *there are constants* $c(\epsilon) > 0$ *with*

$$|a_{d,l}| \leq c(\epsilon) d^{\epsilon}.$$

Further suppose that they have an analytic continuation to the plane $Re(s) > 1 - 1/2k$ *with at most one simple pole at* $s = 1$ *for some* $k \geq 1$.

Assume that

$$\frac{1}{T} \int_{-T}^{T} |F_l(\sigma + it)|^2 dt$$

is bounded for $\sigma \in (\alpha, 1)$ *and* $T \in \mathbb{R}^+$, *if* $\alpha \in (1 - \frac{1}{2k}, 1)$ *is fixed.*
Let $M_1 \subset M_2 \subset \dots$ *be finite sets of primes with* $\mathbb{P} = \bigcup_{j=1}^{\infty} M_j$.
Suppose $\lim_{j->\infty} F_{l,M_j}(s, \theta_j) = f_l(s)$ *uniformly in* $|s - (1 - \frac{1}{4k})| \leq r < \frac{1}{4k}$ *for fixed* $r > 0$.

Then for any $\epsilon > 0$ *there exists a set* $A_\epsilon \subset \mathbb{R}$ *such that for all* $l = 1, \dots, n$ *and all* $t \in A_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})| \leq r-\epsilon} |F_l(s + it) - f_l(s)| < \epsilon$$

and

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0, T))}{T} > 0.$$

Corollary 3.1. *Let*

$$G_m(s) := \frac{\prod_{b=1}^{b=N_m} F_{m,b}(s)}{\prod_{b=1}^{b=N_m^*} F_{m,b}^*(s)} \text{ for } m = 1, \dots, m_0.$$

Suppose that the functions $F_{m,b}(s), F^*_{m,b}(s)$ *satisfy all the conditions of Lemma 3.1 for* $m = 1, \dots, m_0$ *and* $1 \le b \le N_m$ *resp.* $1 \le b \le N^*_m$. *Assume that* $\lim_{j\to\infty} G_{m,M_j}(s,\theta_j) = f_m(s)$ *and* $\lim_{j\to\infty} F_{m,b,M_j}(s,\theta_j) = f_{m,b}(s)$ *uniformly in* $|s - (1-(4k)^{-1})| \le r$. *Under the further conditions that*

$$\max_{m,b,|s|\le r} |f_{m,b}(s)| > 0$$

and

$$f_m(s) = \frac{\prod_{b=1}^{b=N_m} f_{m,b}(s)}{\prod_{b=1}^{b=N_m^*} f_{m,b}^*(s)} \text{ for } |s| \le r$$

we have:
For any $\epsilon > 0$ *there is a set* $B_\epsilon \subset \mathbb{R}$ *such that for all* $m = 1, \dots, m_0$ *and all* $t \in B_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})|\le r-\epsilon} |G_m(s+it) - f_m(s)| < \epsilon$$

and

$$\liminf_{T\longrightarrow\infty} \frac{vol(B_\epsilon \cap (0,T))}{T} > 0.$$

Proof: (of Lemma 3.1)
Notation: $D_{k,r} := \{s \in \mathbb{C} \mid |s-(1-(4k)^{-1})| \le r\}$.
$\|f(s)\|_r := \max_{s\in D_{k,r}} |f(s)|$.

Basically we follow the proof of Voronin [**14**, p.256]:
$F_{l,M_j}(s,\theta)$ depends continuously on the finite vector $(\theta_p)_{p\in M_j}$. Therefore there exists for all $\epsilon > 0$ a $\delta(\epsilon) > 0$ such that

$$\|F_{l,M_j}(s,\theta^{(1)}) - F_{l,M_j}(s,\theta^{(2)})\|_r \le \epsilon$$

if $|\theta_p^{(1)} - \theta_p^{(2)}| \le \delta$ for all $p \in M_j$.

According to the conditions of the Lemma we have for $j \in \mathbb{N}$ large enough and all l:

$$\|F_{l,M_j}(s,\theta_j) - f_l(s)\|_r < \epsilon$$

Therefore $|\theta_p - \theta_{p,j}| < \delta = \delta(j,\epsilon)$ for all $p \in M_j$ implies:

$$\|F_{l,M_j}(s,\theta) - f_l(s)\|_r < 2\epsilon$$

If $\theta(\tau) := \frac{\tau}{2\pi}\big(\log(p)\big)_{p\in\mathbb{P}}, \tau \in \mathbb{R}$, then by the definition of $F_{l,M}(s,\theta)$ we have the equality $F_{l,M}(s,\theta(\tau)) = F_{l,M}(s+i\tau,0)$. The symbol 0 in $F_{l,M}(s+i\tau,0)$ denotes the zero in the space $\mathbb{R}^{\mathbb{P}}$.

Hence if we have for all $p \in M_j$

$$|\tau\frac{\log p}{2\pi} - \theta_{j,p} \mod \mathbb{Z}| < \delta, \tag{1}$$

then

$$\|F_{l,M_j}(s+i\tau,0) - f_l(s)\|_r < 2\epsilon. \tag{2}$$

Let A_δ be the set of all τ satisfying (1) and $T_0 > 1$.

Set $B := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau) - F_{l,M_j}(s+i\tau,0)|^2 d\sigma dt d\tau$

Set $Q := \mathbb{P}\cap(0,z]$, with $z > p$ for all $p \in M_j$. Then

$$B \le 2(S_1+S_2)$$

with

$$S_1 := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_{l,Q}(s+i\tau,0) - F_{l,M_j}(s+i\tau,0)|^2 d\sigma dt d\tau$$

and

$$S_2 := \frac{1}{T}\int\limits_{A_\delta\cap[T_0,T]}\int\limits_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\sigma dt d\tau.$$

To estimate S_1 notice, that
$|F_{l,Q}(s+i\tau,0) - F_{l,M_j}(s+i\tau,0)| = |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|$

Since the numbers $\log(p), p \in \mathbb{P}$ are linearly independent over $\mathbb{Q}$, the curve $\gamma(\tau) := \frac{\tau}{2\pi}(\log(p))_{2\le p\le z}$ is uniformly distributed $\mod \mathbb{Z}$. (Theorem 2.14).

For fixed z and M_j the family of functions $\{g_s\}_{s\in D_{k,r}}$

$$g_s(\theta) := |F_{l,Q}(s,\theta) - F_{l,M_j}(s,\theta)|^2$$

is uniformly bounded and equicontinuous in $(\theta_p)_{p\le z}$, and it depends only on $(\theta_p)_{p\le z} \mod \mathbb{Z}$.

Therefore because of Theorem 2.16

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{A_\delta\cap(T_0,T)} |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|^2 d\tau$$
$$= \int_{\mathcal{D}} |F_{l,Q}(s,\theta) - F_{l,M_j}(s,\theta)|^2 d\theta,$$

uniformly in $s \in D_{k,r}$. We have

$\mathcal{D} = \{(\theta_p)_{p\le z} \mid \forall p \in M_j : |\theta_p - \theta_{j,p} \mod \mathbb{Z}| < \delta \text{ and } \forall p \le z : 0 \le \theta_p \le 1\}$.

Because of $F_{l,Q}(s,\theta) = F_{l,M_j}(s,\theta) F_{l,Q\setminus M_j}(s,\theta)$ and equation (2) we have

$$\int_{\mathcal{D}} |F_{l,Q}(s,\theta(\tau)) - F_{l,M_j}(s,\theta(\tau))|^2 d\theta$$
$$\le (\max_l \|f_l\|_r + 2\epsilon)^2 \int\limits_{\mathcal{D}} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta$$

The functions $F_{l,Q\setminus M_j}(s,\theta) - 1$ do not depend on the variables θ_p for $p \in M_j$. So $\big(d\theta := \prod_{\substack{p\in\mathbb{P}\\ p\le z}} d\theta_p$, $d\theta' := \prod_{p\in Q\setminus M_j} d\theta_p$ and $d\theta'' := \prod_{p\in M_j} d\theta_p\big)$

$$\int\limits_{\mathcal{D}} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta$$
$$\le \int\limits_{\mathcal{D}\cap(\theta_p)_{p\in M_j}} \Big(\int\limits_{\forall_{p\in Q\setminus M_j}:0\le\theta_p\le 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta'\Big) d\theta''$$
$$\le vol(\mathcal{D}) \int\limits_{\forall_{p\in Q\setminus M_j}:0\le\theta_p\le 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta'.$$

Since $F_{l,Q\setminus M_j}(s,\theta) = \prod_{p\in Q\setminus M_j} f_{l,p}\big(p^{-s}\exp(-2\pi i\theta_p)\big)$ we get

$$F_{l,Q\setminus M_j}(s,\theta) - 1 = \sum_{m>1} b_m(\theta)m^{-s}$$

where $b_m(\theta) = \prod_{p\in Q\setminus M_j} a_{p,l}^{(v_p(m))} e^{-2\pi i v_p(m)\theta_p}$ and so

$$|F_{l,Q\setminus M_j}(s,\theta) - 1|^2 = \sum_{m>1} |b_m(\theta)|^2 m^{-2Re(s)} + \sum_{m_1\neq m_2} b_{m_1}(\theta)\overline{b_{m_2}(\theta)}m_1^{-s}\overline{m_2^{-s}}.$$

Both series are absolutely convergent. Therefore the integration may be done term by term. Since the values of the $b_m(\theta)$ depend on θ, the integral over the second series is zero. The first series is independent of θ. Therefore

$$\int_{\forall_{p\in Q\setminus M_j}:0\leq\theta_p\leq 1} |F_{l,Q\setminus M_j}(s,\theta) - 1|^2 d\theta' = \sum_{m>1} |b_m|^2 m^{-2Re(s)}$$

with $|b_m|^2 = \prod_{p\in Q\setminus M_j} |a_{p,l}^{(v_p(m))}|^2$. For an arbitrary small $\epsilon_1 > 0$ one has $|b^{(m)}|^2 \leq c(\epsilon_1)m^{\epsilon_1}$ because of the conditions on $a_{d,l}$ in the Lemma.

Set $\eta := 2r + \frac{1}{2k} - 1$. Then $\eta < 0$ since $r < \frac{1}{4k}$ and $k \geq 1$. Choose numbers $\epsilon_1 > 0$ and $\delta_1 > 0$ such that $\epsilon_2 := \epsilon_1 + \delta_1 + \eta < 0$. If M_j contains all primes smaller than y_j, then

$$\begin{aligned}
& \sum_{m>1} |b^{(m)}|^2 m^{-2Re(s)} \\
\leq\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_1 - 2 + \frac{1}{2k} + 2r} \\
=\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_1 - 1 + \delta_1 + \frac{1}{2k} + 2r} m^{-1-\delta_1} \\
\leq\ & c(\epsilon_1) \sum_{m>y_j} m^{\epsilon_2} m^{-(1+\delta_1)} \\
\leq\ & c(\epsilon_1)\zeta(1+\delta_1)y_j^{\epsilon_2}.
\end{aligned}$$

We have

$$\sum_{m>1} |b^{(m)}|^2 m^{-2Re(s)} \leq c(\epsilon_1)\zeta(1+\delta_1)y_j^{\epsilon_2}.$$

Then (δ_1 and ϵ_2 are fixed):

$$S_1 \leq n(\max_l \|f_l\|_r + 2\epsilon)^2 vol(\mathcal{D}) c(\epsilon_1)\zeta(1+\delta_1) y_j^{\epsilon_2}.$$

We choose a fixed j large enough (the choice of ϵ_2 and δ_1 depends only on r and k) such that

$$4n(\max_l \|f_l\|_r + 2)^2 c(\epsilon_1)\zeta(1+\delta_1)\frac{1}{\epsilon^2} < y_j^{-\epsilon_2}$$

This is possible since $\bigcup_{j=1}^{\infty} M_j = \mathbb{P}$ and $M_j \subset M_{j+1}$ and because we may choose δ sufficiently small such that $\epsilon = \epsilon(\delta, j) \leq 1$ in $\max_l \|f_l\|_r + 2\epsilon$. Then

$$S_1 < 1/4 \quad vol(\mathcal{D})\epsilon^2.$$

From now on j is fixed, thus also $vol(\mathcal{D})$. Now we estimate S_2:

$$\begin{aligned} S_2 &= \frac{1}{T} \int\limits_{A_\delta \cap [T_0,T]} \int\limits_{D_{k,r}} \sum_{l=1}^{n} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\sigma dt d\tau \\ &= \int\limits_{D_{k,r}} \sum_{l=1}^{n} \frac{1}{T} \int\limits_{A_\delta \cap [T_0,T]} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\tau d\sigma dt. \end{aligned}$$

To cancel the pole at $s = 1$ we multiply by $\phi(s) = 1 - 2^{1-s}$. This function has a simple zero at $s = 1$. We get for $1 - \frac{1}{2k} < Re(s) \leq 1 - \frac{1}{4k} + r < 1$:

$0 < a(r) < |\phi(s)| < c(r)$ for some numbers $a(r), c(r) \in \mathbb{R}$.

This implies for $s \in D_{r,k}$:

$$\begin{aligned} &\frac{1}{T}\int_{-T}^{T} |\phi(s+i\tau)F_l(s+i\tau) - \phi(s+i\tau)F_{l,Q}(s+i\tau,0)|^2 d\tau \\ \leq\ & c(r)\frac{1}{T}\int_{-T}^{T} |F_l(s+i\tau) - F_{l,Q}(s+i\tau,0)|^2 d\tau. \end{aligned}$$

Since $\frac{1}{T}\int\limits_{-T}^{T} |F_j(\sigma+it)|^2 dt$ is bounded for $\sigma \in (\alpha, 1)$ with fixed $\alpha > 1 - \frac{1}{2k}$ and $T \in \mathbb{R}^+$, the same applies to the function

$$\phi(s+i\tau)F_l(s+i\tau) - \phi(s+i\tau)F_{l,Q}(s+i\tau,0).$$

So we can apply Theorem 2.11 to get:

$$\lim_{T\longrightarrow\infty}\frac{1}{2T}\int_{-T}^{T}|\phi(s+i\tau)F_l(s+i\tau)-\phi(s+i\tau)F_{l,Q}(s+i\tau,0)|^2d\tau=\sum_{m=1}^{\infty}|c_m|^2m^{-2Re(s)},$$

where $\sum_{m=1}^{\infty}c_m m^{-s}$ $(Re(s)>1)$ is the Dirichlet series of

$$\phi(s)(F_l(s)-F_{l,Q}(s,0)).$$

Therefore we have for z and $T>T(z)$ sufficiently large (remember $Q=\mathbb{P}\cap(0,z]$ and $z>y_j$)

$$\frac{1}{2T}\int_{-T}^{T}|F_l(s+i\tau)-F_{l,Q}(s+i\tau,0)|^2d\tau\leq\frac{1}{8n}vol(\mathcal{D})\epsilon^2,$$

since in this case $c_m=0$ for all m with prime divisors less than z. Then

$$\begin{aligned}S_2 &= \frac{1}{T}\int_{A_\delta\cap[T_0,T]}\int_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau)-F_{l,Q}(s+i\tau,0)|^2d\sigma dtd\tau\\ &\leq 2\int_{D_{k,r}}\frac{1}{8}vol(\mathcal{D})\epsilon^2d\sigma dt\\ &\leq \frac{1}{4}vol(\mathcal{D})\epsilon^2.\end{aligned}$$

This gives $(B\leq 2(S_1+S_2))$ for large T and z

$$B=\frac{1}{T}\int_{A_\delta\cap[T_0,T]}\int_{D_{k,r}}\sum_{l=1}^{n}|F_l(s+i\tau)-F_{l,M_j}(s+i\tau,0)|^2d\sigma dtd\tau\leq vol(\mathcal{D})\epsilon^2.$$

Remember that

$\mathcal{D}=\{(\theta_p)_{p\leq z}\mid\forall p\in M_j:|\theta_p-\theta_{j,p}\mod\mathbb{Z}|<\delta\text{ and }\forall p\leq z:0\leq\theta_p\leq 1\}$

and

$$A_\delta=\{\tau\mid|\tau\frac{\log p}{2\pi}-\theta_{j,p}\mod\mathbb{Z}|<\delta\},$$

where δ depends only on ϵ and j. We get because of Theorem 2.14

$$\lim_{T\to\infty}\frac{vol(A_\delta\cap[T_0,T])}{T}=vol(\mathcal{D})=(2\delta)^{\#M_j}>0. \qquad (3)$$

Then for every T sufficiently large there is a set $Y \subset A_\delta \cap [T_0, T]$ with $vol(Y) > \frac{1}{4} vol(\mathcal{D})T$ and for all $\tau \in Y$:

$$\int\limits_{D_{k,r}} \sum_{l=1}^{n} |F_l(s+i\tau) - F_{l,M_j}(s+i\tau, 0)|^2 d\sigma dt \leq 2\epsilon^2.$$

To see this define $Y := \{\tau \mid |g(\tau)| \leq 2\epsilon^2\} \cap A_\delta \cap [T_0, T]$ with $g(\tau) := \int\limits_{D_{k,r}} \sum\limits_{l=1}^{n} |F_l(s+i\tau) - F_{l,M_j}(s+i\tau, 0)|^2 d\sigma dt$. Denote its complement in $A_\delta \cap [T_0, T]$ by Y^c. Then

$$\frac{2\epsilon^2 vol(Y^c)}{T} \leq \frac{1}{T} \int_{Y^c} |g(\tau)| d\tau \leq \frac{1}{T} \int_{A_\delta \cap [T_0, T]} |g(\tau)| d\tau \leq vol(\mathcal{D})\epsilon^2.$$

Therefore

$$2 \frac{vol(A_\delta \cap [T_0, T]) - vol(Y)}{T} \leq vol(\mathcal{D})$$

and with equation (3) we conclude $vol(Y) > \frac{1}{4} vol(\mathcal{D})T$ for large T. Because of the definition of A_δ we have $\|F_{l,M_j}(s+i\tau, 0) - f_l(s)\|_r < 2\epsilon$. This gives

$$(\int\limits_{D_{k,r}} |F_l(s+i\tau) - f_l(s)|^2 d\sigma dt)^{1/2} \leq 4\epsilon.$$

As both functions F_l and f_l are holomorphic in the interior of $D_{k,r}$ for $s = \sigma + it$, continuous on the border of $D_{k,r}$ and ϵ is arbitrary, the Lemma follows from Theorem 2.9.

□

Recall the definition of $L_M(s, \chi, \theta)$ and $L_p(s, \chi, \theta)$ at the beginning of this chapter.

LEMMA 3.2. *Let $\chi_1, \ldots, \chi_n$ be linearly independent non-Abelian characters of $G := G(K/\mathbb{Q})$, where K is a finite normal algebraic extension of $\mathbb{Q}$. Let $k := \#G$ and $0 < r < \frac{1}{4k}$.*
Suppose that $f_1(s), \ldots, f_n(s)$ are analytic for $|s| < r$ and continuous for $|s| \leq r$ and not zero on the disc $|s| \leq r$. Then for every pair $\epsilon > 0$ and $y \in \mathbb{R}^+$ there exists a finite set of primes M containing all primes smaller than y and $\theta \in \mathbb{R}^{\mathbb{P}}$ such that:

$$\max_{j=1}^{n} \max_{|s| \leq r} |L_M(s + 1 - \frac{1}{4k}, \chi_j, \theta) - f_j(s)| < \epsilon.$$

Proof:

Choose $\gamma > 1$ such that $\gamma^2 r < \frac{1}{4k}$ and

$$\forall_j : \max_{|s|\leq r} |f_j(s) - f_j(s/\gamma^2)| < \epsilon/2.$$

Because $f_j(s) \neq 0$ we can write

$$f_j(\frac{s}{\gamma^2}) = \exp(g_j(s)) \text{ for some } g_j(s) \text{ analytic in } |s| < \gamma^2 r.$$

Hence it is sufficient to prove the Lemma for the logarithms of the functions.
Remember that the Euler-factors (all but finitely many) of Artin L-series $L(s, \chi_j)$ are defined by $1/\det(E_{k_j} - \rho_j(\sigma_p)p^{-s})$, where σ_p is one of the conjugate Frobenius-Automorphisms over $p \in \mathbb{P}$ and
$\rho_j : G \to GL_{k_j}(\mathbb{C})$ is a representation of G with $\chi_j(\sigma) = \mathrm{trace}(\rho_{\mathrm{j}}(\sigma))$ for $\sigma \in G$.
For the Euler-factors of $L_M(s', \chi_j, \theta)$ we get:

$$\log L_p(s', \chi_j, \theta) = \frac{\mathrm{trace}(\rho_{\mathrm{j}}(\sigma_{\mathrm{p}}))\exp(-2\pi \mathrm{i}\theta_{\mathrm{p}})}{\mathrm{p}^{\mathrm{s}'}} + \sum_{m\geq 2} a_{m,p}p^{-ms'}.$$

The first term is equal to $\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}}$. Therefore

$$\log L_M(s', \chi_j, \theta) = \sum_{p\in M} \frac{\chi_j(\sigma_p)e^{(-2\pi i\theta_p)}}{p^{s'}} + \sum_{p\in M}\sum_{m\geq 2} a_{m,p}p^{-ms'}.$$

The second term is a uniformly and absolutely convergent series for all primes in $\mathbb{Q}$, since its coefficients are dominated by the coefficients of $\chi_j(1)\log\zeta(s)$ as remarked on page 9.

We define a real Hilbert space $\mathcal{H}_n^{(R)}$ of vectors of functions holomorphic on the disc $|s| < R$. The scalar product is (always $R' < R$)

$$\langle (h_j)_{j=1}^n, (f_j)_{j=1}^n \rangle := \lim_{R'\to R} Re \int\limits_{|s|\leq R'} \sum_{j=1}^n f_j(s)\overline{h_j(s)}d\sigma dt.$$

The functions h_j and f_j, $j = 1, \dots n$ are holomorphic in $|s| < R$ and satisfy (setting $g := h_j$ or $g := f_j$),

$$\lim_{R'\to R} \int\limits_{|s|\leq R'} |g(s)|^2 d\sigma dt < \infty.$$

This Hilbert space is n times the product of the Hilbert space $\mathcal{H}_2$ (Def. 5, p.13).

Set $R := \gamma r$ $(\gamma > 1)$ and $\eta_p(s) := \left(\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}}\right)_{j=1}^{n}$, where $s' = s + 1 - \frac{1}{4k}$ with $|s| \leq R$.

Denote the different conjugacy classes of the group G by $C_1, \dots, C_N$. Obviously $n \leq N$, since N is the dimension of the vector space of class functions on G.

Denote the different prime classes by $\mathbb{P}_j := \{p \mid \sigma_p \in C_j\}$.
To define θ: In the natural order of each set $\mathbb{P}_j \subset \mathbb{Z}$ denote the primes $p \in \mathbb{P}_j$ by $p_{j,l}$ such that $p_{j,1} < p_{j,2} < p_{j,3} \ldots < p_{j,l} < p_{j,l+1} < \ldots$.
Set $\theta_{p_{j,l}} := \frac{l}{4}$. Thereby θ_p is defined for all but finitely many primes $p \in \mathbb{P}$. For the primes ramified in K set $\theta_p := 0$.

We will use Theorem 2.6 on conditionally convergent series in real Hilbert spaces.

We only need to show that the series $\eta_p, p \in \mathbb{P}$ fulfills the conditions of this Theorem:

$$\sum_{p\in\mathbb{P}} \|\eta_p\|^2 \leq C \sum_{p\in\mathbb{P}} p^{\frac{1}{2k}-2+2R} < \infty \text{ with } C = n \max_{j=1}^{n}\{\chi_j(1)^2\}.$$

(obviously $\frac{1}{2k} - 2 + 2R < -1$)
For e (as in Theorem 2.6) we can choose any $\varphi(s) \in \mathcal{H}_n^R$ with $\|\varphi\| := \langle\varphi,\varphi\rangle^{1/2} = 1$.
Now we have to show that

$$\sum_{p\in\mathbb{P}} \langle\eta_p,\varphi\rangle$$

is conditionally convergent, or equivalently:
$\lim_{p\to\infty} \langle\eta_p,\varphi\rangle = 0$ and there exist two sets of primes $\mathbb{P}_+$ and $\mathbb{P}_-$ such that $\forall_{p\in\mathbb{P}_+} : \langle\eta_p,\varphi\rangle > 0$, $\sum_{p\in\mathbb{P}_+} \langle\eta_p,\varphi\rangle = \infty$, and

$\forall_{p\in\mathbb{P}_-} : \langle\eta_p,\varphi\rangle < 0$, $\sum_{p\in\mathbb{P}_-} \langle\eta_p,\varphi\rangle = -\infty$.

We compute:

$$\begin{aligned}
\langle \eta_p, \varphi \rangle &= \lim_{R' \to R} Re \int\limits_{|s| \leq R'} \sum_{j=1}^{n} \eta_{p,j}(s) \overline{\varphi_j(s)} d\sigma dt \\
&= \lim_{R' \to R} Re \int\limits_{|s| \leq R'} \sum_{j=1}^{n} \chi_j(\sigma_p) e^{-2\pi i \theta_p} p^{-s'} \overline{\varphi_j(s)} d\sigma dt \\
&= \lim_{R' \to R} Re \big(e^{-2\pi i \theta_p} \int\limits_{|s| \leq R'} p^{-(s+1-\frac{1}{4k})} \big(\sum_{j=1}^{n} \chi_j(\sigma_p) \overline{\varphi_j(s)} \big) d\sigma dt \big).
\end{aligned}$$

It follows that

$$\lim_{p \to \infty} |\langle \eta_p, \varphi \rangle| = 0.$$

Since the characters χ_j are linearly independent and $\varphi \neq 0$, there is a conjugacy class C_l in G such that $\varphi_0(s) := \sum\limits_{j=1}^{n} \chi_j(\sigma_p) \overline{\varphi_j(s)} \not\equiv 0$ for all $\sigma_p \in C_l$.

As the functions φ_j are holomorphic in the disc $|s| < R$, we have

$$\varphi_0(s) = \sum_{m=0}^{\infty} \overline{\alpha_m s^m}.$$

For $p \in C_l$ we get

$$\begin{aligned}
\langle \eta_p, \varphi \rangle &= \lim_{R' \to R} Re \big(e^{-2\pi i \theta_p} \int\limits_{|s| \leq R'} \exp\big(-\log(p)(s+1-\frac{1}{4k}) \big) \varphi_0(s) d\sigma dt \\
&= Re \big(e^{-2\pi i \theta_p} \Delta(\log p) \big).
\end{aligned}$$

Here $\Delta(x) := \lim\limits_{R' \to R} \int\limits_{|s| \leq R'} \exp\big(-x(s+1-\frac{1}{4k}) \big) \varphi_0(s) d\sigma dt.$

Therefore

$$\begin{aligned}\Delta(x) &= \exp\big(-x(1-\frac{1}{4k})\big)\lim_{R'\to R}\int\limits_{|s|\le R'}\exp(-xs)\varphi_0(s)d\sigma dt\\ &= \pi R^2\exp\big(-x(1-\frac{1}{4k})\big)\sum_{m=0}^{\infty}\frac{(-1)^m\overline{\alpha}_m(xR^2)^m}{(m+1)!}.\end{aligned}$$

We have

$$\|\varphi_0\|^2 = \lim_{R'\to R}\int\limits_{|s|\le R'}|\varphi_0|^2 d\sigma dt = \pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1}.$$

Using the continuous linear mapping $L((f_j)_{j=1}^n) := \sum_{j=1}^n \overline{\chi_j(C_l)}f_j$ we get

$$\|\varphi_0\|^2 = \|L(\varphi)\|^2 \le \|L\|^2\|\varphi\|^2 = \|L\|^2 < \infty.$$

This gives:

$$\pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1} = \|\varphi_0\|^2 \le \|L\|^2.$$

Setting $\beta_m := (-1)^m R^m\overline{\alpha}_m/(m+1)$ we get $\sum_{m=0}^{\infty}|\beta_m|^2 \le \|L\|^2/(\pi R^2)$, which gives us an upper bound for all $|\beta_m|$.
Set

$$F(u) := \sum_{m=0}^{\infty}\frac{\beta_m}{m!}u^m.$$

$F(u)$ is an entire function and $F\not\equiv 0$ since $\varphi_0\neq 0$. For any $\delta>0$ there is a sequence of positive real numbers with $u_n \longrightarrow \infty$ such that

$$|F(u_n)| > \exp\big(-(1+2\delta)u_n\big).$$

This is a consequence of Corollary 2.2. We have
$\Delta(x) = \pi R^2\exp\big(-x(1-\frac{1}{4k})\big)F(xR)$. Set $x_n := u_n/R$. Then

$$|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)$$

for $\delta_0>0$ sufficiently small and x_n sufficiently large.

As a consequence we find subintervals I_n of $[x_n - 1, x_n + 1]$ of length greater than $\frac{1}{2x_n^8}$ in which one of the inequalities

$$|Re\Delta(x)| > \frac{e^{-(1-\delta_0)x}}{200} \text{ or} \tag{4}$$

$$|Im\Delta(x)| > \frac{e^{-(1-\delta_0)x}}{200} \tag{5}$$

holds.

To prove this we approximate Δ by polynomials. Set $N := [x_n] + 1$. Let B be an upper bound for the $|\beta_m|$. This gives $|F(xR)| \leq Be^{xR}$. For $x \in [x_n - 1, x_n + 1]$ we have (remember $R < \gamma^2 r < 1/4k$)

$$\begin{aligned} |\sum_{m=N^2}^{\infty} \frac{\beta_m}{m!}(xR)^m| &\leq B \sum_{m=N^2}^{\infty} \frac{1}{m!}(xR)^m \leq B\frac{(xR)^{N^2}}{N^2!} \sum_{m=0}^{\infty} \frac{1}{m!}(xR)^m \\ &\leq B\frac{N^{N^2}}{N^2!}e^N \leq B\Big(\frac{N}{N^2/e}\Big)^{N^2} e^N \leq B\frac{e^{N^2+N}}{N^{N^2}} \leq e^{-2x_n} \end{aligned}$$

if x_n is sufficiently large.
For $x \in [x_n - 1, x_n + 1]$ we also have $\big((1 - \frac{1}{4k}) < 1\ \big)$.

$$\sum_{N^2=m}^{\infty} \frac{\big(-(1-\frac{1}{4k})x\big)^m}{m!} \leq e^{-2x_n}$$

Hence $F(xR) = P_1(x) + O(e^{-2x_n})$ and $\exp(-(1 - \frac{1}{4k})x) = P_2(x) + O(e^{-2x_n})$, where P_1 and P_2 are polynomials of degree $N^2 - 1$. This gives $\Delta(x) = P_n(x) + O(e^{-x_n})$ for all $N = [x_n] + 1$ and $x \in [x_n - 1, x_n + 1]$, where $P_n(x)$ is a polynomial of degree less than N^4.

Thus we also have $Re\Delta(x) = Re(P_n(x)) + O(e^{-x_n})$ and $Im\Delta(x) = Im(P_n(x)) + O(e^{-x_n})$. However if $x \in \mathbb{R}$, then $Re(P_n(x))$ and $Im(P_n(x))$ are polynomials with real coefficients.
We may suppose that either $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$ or $|Im\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$, since $|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)$.

Suppose that $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$. Denote the polynomial $Re(P_n(x))$ again by $P_n(x)$. Since $|Re\Delta(x_n)| > \frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$ we have $\frac{1}{4}e^{-(1-\delta_0)x_n} \leq |P_n(x_n)|$ for large n. Set $a := \max_{|x-x_n|\leq 1} |P_n(x)|$. Then there exists a $\xi \in [x_n - 1, x_n + 1]$ such that $a = |P_n(\xi)|$. There exists a $\kappa \in (\xi, x)$ or $\kappa \in (x, \xi)$ such that
$|P_n(\xi) - P_n(x)| = |P_n'(\kappa)(x - \xi)|$. Set $\tau := N^8|\xi - x|$. Then because of Theorem 2.8 we have $|P_n(\xi) - P_n(x)| \leq \tau a$. If $\tau \leq 1/2$ then

$|1-\frac{P_n(x)}{P_n(\xi)}|\le 1/2$, therefore $|P_n(x)|\ge \frac{a}{2}\ge \frac{|P_n(x_n)|}{2}\ge \frac{1}{8}e^{-(1-\delta_0)x_n}$ for all x with $|x-\xi|\le \frac{1}{2N^8}$. It follows that

$$|Re\Delta(x)|\ge \frac{1}{16}e^{-(1-\delta_0)x_n}\ge \frac{1}{16e^2}e^{-(1-\delta_0)x}\ge \frac{1}{200}e^{-(1-\delta_0)x}$$

for large n and $|x-\xi|\le \frac{1}{2N^8}$.

The same argumentation applies to $Im\Delta(x)$ if $|Im\Delta(x_n)|>\frac{1}{2}\exp\big(-(1-\delta_0)x_n\big)$.

In the natural order of the set $\mathbb{P}_l$ we have for $p_r\in\mathbb{P}_l$, and $p_1<p_2<\ldots<p_r<\ldots$ that $\theta_{p_r}=r/4$ by the definition of θ. Thus we get $e^{-2\pi i\theta_{p_r}}=(-i)^r$. Therefore

$$\langle \eta_{p_r},\varphi\rangle = Re\big((-i)^r\Delta(\log(p_r))\big).$$

One of the inequalities (4), (5) is satisfied infinitely often. Consider the interval $I_n:=[\alpha,\alpha+\beta]$ such that on I_n one of the inequalities $|Im(\Delta(x))|\ge \frac{1}{200}e^{-(1-\delta_0)x}$ or $|Re(\Delta(x))|\ge \frac{1}{200}e^{-(1-\delta_0)x}$ holds and $\beta\ge \frac{1}{2x_n^8}$.

According to Theorem 2.5 the number of primes $p\in\mathbb{P}_l$ for which $\log p\in I_n$ is ($h_l:=\#C_l$):

$$\begin{aligned}\pi(e^{\alpha+\beta},C_l)-\pi(e^\alpha,C_l) &= \frac{h_l}{k}\int_{e^\alpha}^{e^{\alpha+\beta}}\frac{dt}{\log t}+O(e^{\alpha+\beta}e^{-a\alpha^{1/2}})\\ &\ge \frac{h_l}{k}e^\alpha\left(\frac{e^\beta-1}{\alpha+\beta}+O(\frac{e^\beta}{e^{a\alpha^{1/2}}})\right).\end{aligned}$$

Since $2\ge\beta\ge\frac{1}{2x_n^8}$, we get $e^\beta-1\ge\frac{1}{2x_n^8}$ and $\frac{e^\beta-1}{\alpha+\beta}\ge\frac{e^\beta-1}{x_n+2}\ge\frac{1}{2x_n^9+4x_n^8}$. Next $\frac{e^\beta}{e^{a\alpha^{1/2}}}\le\frac{e^2}{e^{a\sqrt{x_n-1}}}$ and $e^\alpha\ge e^{x_n}/e$. Thus for x_n sufficiently large we get

$$\pi(e^{\alpha+\beta},C_l)-\pi(e^\alpha,C_l)\ge\frac{h_l}{k}\frac{e^{x_n}}{x_n^{10}}.$$

The number of primes p with $\log p\in I_n$ and $\exp(-2\pi i\theta_p)=1$, $\exp(-2\pi i\theta_p)=-1, \exp(-2\pi i\theta_p)=i$, or $\exp(-2\pi i\theta_p)=-i$ is therefore greater than $\frac{h_l}{k}\frac{e^{x_n}}{4x_n^{10}}$.

Therefore

$$\sum_{\substack{p\in\mathbb{P}_l,\log p\in I_n\\ \mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p))>0}}\langle\eta_p,\varphi\rangle > c_1e^{\delta_0x_n/2}$$

for some positive constant c_1. The same holds for a subset of primes with $\mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p)) < 0$. The sum is less than $-c_1 e^{\delta_0 x_n/2}$. As $x_n \to \infty$ the corresponding series diverge to $+\infty$ and $-\infty$.

The rest of the proof is a consequence of Corollary 2.3: $R/\gamma = r < R$. According to Theorem 2.6 we can order $\mathbb{P}$ such that we get a sequence of finite subsets $M_n \subset \mathbb{P}$ with $M_n \subset M_{n+1}$, $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{P}$ and uniformly in $|s| \leq r$ $\lim_{n\to\infty} \log L_{M_n}(z, \chi_j, \theta) = g_j(s)$ for $z = s+1-\frac{1}{4k}$. Therefore

$$|f_j(s)-L_M(s+1-\frac{1}{4k}, \chi_j, \theta)| \leq |f_j(s)-f_j(s/\gamma^2))|+|e^{g_j(s)}-L_M(z, \chi_j, \theta)| < \epsilon$$

for some $n \in \mathbb{N}$ sufficiently large, $|s| \leq r$ and $M := M_n$. Because of $\bigcup_{n\in\mathbb{N}} M_n = \mathbb{P}$ we may choose $n \in \mathbb{N}$ such that all primes less than a given $y \in \mathbb{R}_+$ are contained in $\mathbb{P}$. □

REMARK 3.1. *In the preceeding Lemma we may replace the set* $\mathbb{P}$ *by* $\mathbb{P} \setminus \{p_1, \ldots, p_d\}$, *where* $p_1, \ldots, p_d$ *are primes. The set* M *may be replaced by a finite set of primes* $M \subset \mathbb{P} \setminus \{p_1, \ldots, p_d\}$ *containing all primes smaller than* y. *Also we may replace for a finite number of primes the factors* $L_p(s, \chi)$ *by different Euler-factors satisfying the conditions of Lemma 3.1 and its Corollary 3.1.*

The proof of the Remark is obvious because of the proof of the Lemma, since it was proved that the series η_p is conditionally convergent and this persists if we only change a finite number of the η_p.

CHAPTER 4

A Mean Value Theorem

THEOREM 4.1. *Assume that a Dirichlet series* $\sum_{n=1}^{\infty} a_n n^{-s}$ *satisfies* $a_n = O_\epsilon(n^\epsilon)$ *for every* $\epsilon > 0$. *Suppose that this series converges for* $Re(s) > 1$ *absolutely and can be analytically continued to the complex plane and has no pole except a simple pole at* $s = 1$. *Denote this function by* $f(s)$. *Suppose further that* $|f(s)|^2 = O(|t|^M)$ *for some* $M := M(a_0, b_0) \in \mathbb{R}$ *and* $s = \sigma + it$ *where* $|t| \geq 1$ *and* $\sigma \in [a_0, b_0]$ *with* $a_0, b_0 \in \mathbb{R}$ *and* $a_0 < 0$, $b_0 > 1$. *Then*

$$\frac{1}{T}\int_{-T}^{T} |f(s+it)|^2 dt$$

is bounded for every s *with* $Re(s) > \max\{1 - \frac{1}{M+1}, 1/2\}$. *We can choose* $M = \inf\{m : |f(s)|^2 = O(|t|^m)\}$.

<u>Proof</u>: Obviously there is a $\xi > 0$ such that

$$\frac{1}{T}\int_{-T}^{T} |f(s+it)|^2 dt = O(T^\xi)$$

(take for example $\xi := M$).

Set $\mu := \inf\{M : |f(s)|^2 = O(|t|^M)\}$.

Using Lemma 2.1, we get for $Re(s) > 1$, $(\delta > 0, c > 1, c > \sigma)$

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-\delta n} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw.$$

Because of the condition $a_n = O_\epsilon(n^\epsilon)$ the series on the left side of the equation is absolutely convergent for all $Re(s) > 0$ and therefore it is

a holomorphic function in this plane. Using Stirling's formula on the Γ-function we get $|\Gamma(s)| \leq C_{[a,b]}|t|^{\sigma-1/2}\exp(-\frac{\pi}{2}|t|)$, where $s = \sigma + it$ and $\sigma \in [a,b]$ for every interval $[a,b]$.
Therefore and because of $|f(s)|^2 = O(|t|^M)$ the function
$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw$ is an analytic function for all $c > 0$ and $b_0 > Re(s) > 0$. If $\sigma > \alpha > \sigma - 1$, we have

$$\frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw =$$
$$\frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw + f(s) + \mathrm{Res}_{w=1}\Gamma(w-s)f(w)\delta^{s-w}$$

because of the Residue Theorem. Set $B := \mathrm{Res}_{s=1}f(s)$. Then we find for f the expression

$$f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-\delta n} - \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw - B\Gamma(1-s)\delta^{s-1},$$

where $Re(s) \geq 1/2$, $\sigma > \alpha > \sigma - 1$.

Set $Z_1 := \sum\limits_{n=1}^{\infty}\frac{a_n}{n^s}e^{-\delta n}$ and $Z_2 := \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw$.

We have $Z_3 := B\Gamma(1-s)\delta^{s-1} = O(|t|^{1-\sigma-1/2}e^{-\frac{\pi}{2}|t|}\delta^{\sigma-1})$. This implies $B\Gamma(1-s)\delta^{s-1} = O(\delta^{\sigma-1}e^{-\frac{\pi}{2}|t|})$, if $|t| \geq 1$ and $1/2 \leq \sigma \leq 1$.
For $x, y \in \mathbb{C}$ we have $|x+y|^2 \leq 2(|x|^2 + |y|^2)$, therefore
$|Z_1 + Z_2 + Z_3|^2 \leq 4(|Z_1|^2 + |Z_2|^2 + |Z_3|^2)$.
If $\sigma \geq a > 1/2$, then using Lemma 2.2, we get

$$\begin{aligned}\int\limits_{T/2}^{T}|Z_1|^2dt &= O(T\sum_{m=1}^{\infty}\frac{|a_m|^2}{m^{2a}}e^{-2\delta m}) + O\left(\sum_{m\neq n}\frac{|a_m||a_n|e^{-(m+n)\delta}}{m^\sigma n^\sigma|\log(m/n)|}\right)\\ &= O_a(T) + O(\delta^{2\sigma-2-\epsilon})\end{aligned}$$

for some small $\epsilon > 0$ $\big($since $a_n = O(n^\epsilon)\big)$.

Set $w := \alpha + iv$. We obtain

$$
\begin{aligned}
|Z_2| &\le \frac{\delta^{\sigma-\alpha}}{2\pi}\int_{-\infty}^{\infty} |\Gamma(w-s)f(s)|dv \\
&\le \frac{\delta^{\sigma-\alpha}}{2\pi}\Big(\int_{-\infty}^{\infty} |\Gamma(w-s)|dv \int_{-\infty}^{\infty} |\Gamma(w-s)f^2(w)|dv\Big)^{1/2}.
\end{aligned}
$$

Since the first integral is just an integral over the Γ-function, it is bounded. Assume $T \geq |t|$ (recall that $s = \sigma + it$). Set $I_T := (-\infty, 2T] \cup [2T, \infty)$:

$$
\int_{I_T} |\Gamma(w-s)f^2(w)|dv = O\Big(\int_{I_T} e^{-\frac{\pi}{2}|v-t|}|v-t|^{-1/2}|v|^M dv\Big) = O\big(e^{-\frac{\pi}{3}T}\big).
$$

Hence

$$
\begin{aligned}
\int_{T/2}^{T} |Z_2|^2 dt &= O\Big(\delta^{2\sigma-2\alpha}\frac{T}{2}O(e^{-\frac{\pi}{3}T}) + \delta^{2\sigma-2\alpha}\int_{-2T}^{2T} |f(w)|^2 \big(\int_{T/2}^{T} |\Gamma(w-s)|dt\big)dv\Big) \\
&= O(\delta^{2\sigma-2\alpha}) + O\Big(\delta^{2\sigma-2\alpha}\int_{-2T}^{2T} |f(w)|^2 dv\Big) = O\big(\delta^{2\sigma-2\alpha}T^{1+M}\big).
\end{aligned}
$$

For Z_3 we get

$$
\int_{T/2}^{T} |Z_3|^2 dt = O\bigg(\delta^{2(\sigma-1)}\int_{T/2}^{T} \exp(-\frac{2\pi}{2}|t|)dt\bigg) = O(\delta^{2(\sigma-1)}).
$$

This gives ($M = \mu + \epsilon$):

$$
\int_{T/2}^{T} |f(s)|^2 dt = O_a(T) + O(\delta^{2\sigma-2-\epsilon}) + O(\delta^{2\sigma-2\alpha}T^{1+\mu+\epsilon}) + O(\delta^{2(\sigma-1)}).
$$

Set $\delta := T^{-\frac{\gamma}{2}}$ with $\gamma := \frac{\epsilon+\mu}{1-\alpha}$. Then $\gamma > 0$ and $\delta > 0$ is well defined. For $\sigma > \max\{1 - \frac{1-\alpha}{\mu+1+\epsilon}, a, 1/2\}$ we get

$\delta^{2(\sigma-2)-\epsilon} = O(T)$, $\delta^{2(\sigma-2)} = O(T)$ and $\delta^{2\sigma-2\alpha}T^{1+\mu+\epsilon} = O(T)$.
Taking the limits $\alpha \to 0$ and $\epsilon \to 0$, we get

$$\int_{T/2}^{T} |f(s)|^2 dt = O_a(T)$$

for $\sigma > \max\{1 - \frac{1}{\mu+1}, a\}$.

Adding up $\int_{T/2}^{T} |f(s)|^2 dt + \int_{T/4}^{T/2} |f(s)|^2 dt + \int_{T/8}^{T/4} |f(s)|^2 dt + \ldots$ gives $\int_{1}^{T} |f(s)|^2 dt = O_a(T)$ and analogously $\int_{-T}^{1} |f(s)|^2 dt = O_a(T)$ for the fixed $a > 1/2$.

Since $a > 1/2$ can be chosen arbitrary, we have
$Re(s) > \max\{1 - \frac{1}{\mu+1}, 1/2\}$ as a sufficient condition for
$\frac{1}{T} \int_{-T}^{T} |f(s+it)|^2 dt$ to be bounded. □

REMARK 4.1. For Hecke L-series over a field k with $\mathbb{Q} \subset k \subset K$, where K is a finite normal extension of $\mathbb{Q}$, the conditions of Theorem 4.1 are satisfied with $M = [K : \mathbb{Q}]$.

Proof: Denote the Dirichlet-coefficients of the Hecke-L-series $L(s, \chi)$ by $a_n(\chi)$ and the Dirichlet coefficients of the Dedekind Zeta-function $\zeta_k(s)$ by a_n. Then we have $|a_n(\chi)| \le a_n$, where a_n is the number of ideals of norm n in the ring of integers of k. Therefore we have $|a_n| = O_\epsilon(n^\epsilon)$ [**22**, p.152].

Every Hecke L-series satisfies a functional equation.
$\Lambda(s, \chi) := C^s \Gamma(\frac{s+1}{2})^a \Gamma(\frac{s}{2})^{r_1 - a} \Gamma(s)^{r_2} L(s, \chi)$,
where r_1 is the number of real embeddings of k, r_2 the number of complex embeddings of k, a is the number of infinite places of the conductor of χ and $C \in \mathbb{R}^{>0}$ is a constant. Then $r_1 + 2r_2 = [k : \mathbb{Q}] \le [K : \mathbb{Q}]$. We have $\Lambda(s, \chi) = W\Lambda(1 - s, \chi)$, where W is a root of unity. $L(s, \chi)$ is a holomorphic function for all $s \in \mathbb{C}$ if $L(s, \chi) \neq \zeta_k(s)$. If $L(s, \chi) = \zeta_k(s)$ there is a simple pole at $s = 1$.

According to a Theorem of Lavrik [**19**, (p.133: Lemma 2.1)] we have:

$\Lambda(s,\chi) = \frac{c}{s(1-s)} + \sum_{n=1}^{\infty}(a_n f(\frac{C}{n},s) + W\bar{a}_n f(\frac{C}{n},1-s))$, where c is a constant for ζ_k and zero in all other cases.
We have $f(x,s) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} x^z\Gamma(\frac{z+1}{2})^a\Gamma(\frac{z}{2})^{r_1-a}\Gamma(z)^{r_2}\frac{dz}{z-s}$, where $\delta\in\mathbb{R}$ and $\delta>\max\{\mathrm{Re}(s),0\}$. If we take $\delta>\max\{\mathrm{Re}(s)+1,0\}$, then
$|f(x,s)|\leq\frac{x^\delta}{2\pi}\int_{-\infty}^{\infty}|\Gamma(\frac{\delta+it+1}{2})|^a|\Gamma(\frac{\delta+it}{2})|^{r_1-a}|\Gamma(\delta+it)|^{r_2}dt = C_\delta x^\delta$.
This means for $Re(s)\in[-1,2]$ that for some constant C'_δ and $\delta>3$ we have $|\Lambda(s,\chi)|\leq C'_\delta 2\sum_{n\in\mathbb{N}}|a_n(\chi)|\frac{1}{n^\delta}$.
Therefore $|\Lambda(s,\chi)|\leq 2C^4C'_4\zeta_k(4)$. The same holds for ζ_k if we suppose that $|Im(s)|$ is large enough, such that we can ignore $\frac{c}{s(1-s)}$. Because of the well known properties of the Γ-function we therefore get $L(s,\chi) = O(\exp(A|t|)$ and $\zeta_k(s) = O(\exp(A|t|)$ for every fixed strip $Re(s)\in[a,b]$, $Im(s)=t$ and some $A\in\mathbb{R}^{>0}$. To apply the Phragmen-Lindelöf-principle 2.12, we must show that $L(s,\chi) = O(|t|^M)$ on the borders $Re(s)=-\epsilon$ and $Re(s)=1+\epsilon$ for large $t=Im(s)$ and every fixed small $\epsilon>0$. This would imply that $L(s,\chi)=O(|t|^M)$ for all $Re(s)\in[-\epsilon,1+\epsilon]$ and $|Im(s)|=|t|>1$.

The series $L(s,\chi)$ and $\zeta_k(s)$ converge absolutely for all s with $Re(s)=1+\epsilon$ and we have $|L(s,\chi)|\leq\zeta_k(1+\epsilon)$ and $|\zeta_k(s)|\leq\zeta_k(1+\epsilon)$. This is an absolute constant independent of $Im(s)=t$. Using the functional equation we find that $|L(s,\chi)| = O_\epsilon\big(g(|t|)\big)$ and $|\zeta_k(s)| = O_\epsilon\big(g(|t|)\big)$ for s with $Re(s)=-\epsilon$, where

$$g(|t|) = \frac{|\Gamma(\frac{1-s+1}{2})^a\Gamma(\frac{1-s}{2})^{r_1-a}\Gamma(1-s)^{r_2}|}{|\Gamma(\frac{s+1}{2})^a\Gamma(\frac{s}{2})^{r_1-a}\Gamma(s)^{r_2}|}.$$

Stirling's formula gives $|\Gamma(s)| = O\big(|t|^{\sigma-1/2}\exp(-\frac{\pi}{2}|t|)\big)$, where the constant in the big O depends only on the interval $\sigma\in[a,b]$ with $s=\sigma+it$. Therefore $g(|t|) = O\big(|t|^{r_1\frac{1-2\sigma}{2}}|t|^{r_2(1-2\sigma)}\big) = O\big(|t|^{(1-2\sigma)[k:\mathbb{Q}]/2}\big)$ follows. We have $Re(s)=-\epsilon$. Thus we get in the strip $\sigma\in[-\epsilon,1+\epsilon]$ $L(s,\chi)=O(|t|^{M_\epsilon})$ and $\zeta_k(s)=O(|t|^{M_\epsilon})$ with $M_\epsilon=(1+2\epsilon)\frac{[k:\mathbb{Q}]}{2}$. The infimum is obviously $[k:\mathbb{Q}]/2$. □

CHAPTER 5

Main Theorem

We prove the following statement on Artin L-functions over $\mathbb{Q}$:

THEOREM 5.1. *Let K be a finite Galois-extension of $\mathbb{Q}$ and $\chi_1, \dots, \chi_n$ linearly independent characters of the group $G := G(K/\mathbb{Q})$. Let $k := \#G$ and $f_1(s), \dots, f_n(s)$ be holomorphic functions on $|s| < r$ and continuous on $|s| \le r$, where r is a fixed number with $0 < r < \frac{1}{4k}$. Further suppose $f_j(s) \ne 0$ on $|s| \le r$.*

Then for every $\epsilon > 0$ there is a set $A_\epsilon \subset \mathbb{R}$ such that

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$$

and for $j = 1, \dots, n$

$$\forall_{t \in A_\epsilon} \forall_{|s| \le r} : |L(s + 1 - \frac{1}{4k} + it, \chi_j, K/\mathbb{Q}) - f_j(s)| < \epsilon,$$

where $L(z, \chi_j, K/\mathbb{Q})$ denotes the Artin L-function corresponding to the non-Abelian character χ_j.

Proof: The Theorem 2.3 of Brauer states that every character is a finite linear combination $\chi = \sum_l n_l \varphi_l^* - \sum_l m_l \psi_l^*$, where φ_l^*, and ψ_l^* are induced from characters φ_l, ψ_l of degree 1 of subgroups of G. According to Theorem 2.4 we get that $L(z, \chi, K/\mathbb{Q}) = \prod_{l=1}^{m_1} L(z, \varphi_l)^{n_l} \Big/ \prod_{l=1}^{n_1} L(z, \psi_l)^{m_l}$, where the series $L(z, \varphi_l)$ and $L(z, \psi_l)$ are Hecke-L-series over number fields contained in K. These are entire functions with the only exception of the Dedekind ζ-functions which have a simple pole at $z = 1$. Therefore one of the conditions of Lemma 3.1 is satisfied by Remark 4.1: The mean values $\frac{1}{T} \int_{-T}^{T} |f(\sigma + it)|^2 dt$ are bounded even for $\sigma > 1 - \frac{1}{k+1}$, where $k = [K : \mathbb{Q}]$ and $f(z)$ is a Hecke L-function of a number field contained in K. Obviously $1 - \frac{1}{k+1} \le 1 - \frac{1}{2k}$. For the Dirichlet-coefficients

$a_n(\chi)$ of Hecke L-functions we have: $|a_n(\chi)| = O_\epsilon(n^\epsilon)$.
We have to show that the conditions in Corollary 3.1 are fulfilled.

We notice Theorem 2.4 and its Remark. If the characters $\chi_1, \ldots, \chi_n$ are not yet a basis of the class functions of G, then add some more characters (for example from the set of irreducible characters of G). Choose additional holomorphic functions f_j, for example constants $\neq 0$, which then satisfy the conditions of Lemma 3.2.
As we now have a basis of class functions, every character χ_l^*, ψ_l^* can be expressed as a linear combination of this basis.

Choose $\gamma > 1$ such that $\gamma^2 r < \frac{1}{4k}$ and $\max_{|s|\le r} |f_j(s) - f_j(\frac{s}{\gamma^2})| < \epsilon/2$ for $j = 1, \ldots, n$. Apply Lemma 3.2 for the functions $f_j(\frac{s}{\gamma^2})$ and $|s| \le r\gamma$. Now choose a sequence $\epsilon_m := 1/m$, $y_m := \max M_{m-1} + 1$ ($y_0 := 1$), $\theta_m = (\theta_{m,p})_{p\in\mathbb{P}} \in \mathbb{R}^{\mathbb{P}}$ and $M_m \subset \mathbb{P}$ such that Lemma 3.2 with $\epsilon = \epsilon_m$, $y = y_m$ and $M = M_m$ is satisfied. $M_m \subset M_{m+1}$ is a consequence.
The series expansion of the logarithm gives

$$L_{M_m}(s + 1 - \frac{1}{4k}, \theta_m, \chi_j) =$$
$$\sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_m, \kappa\ge 2} a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})}.$$

Then because of $\lim_{m\to\infty} L_{M_m}(s + 1 - \frac{1}{4k}, \theta_m, \chi_j) = f_j(\frac{s}{\gamma^2})$ uniformly in $|s| \le r\gamma$ and $f_j(\frac{s}{\gamma^2}) \neq 0$, we get for the logarithms of these functions:

$$\lim_{m\to\infty}\left(\sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_m,\kappa\ge 2} a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})}\right) = \log f_j(\frac{s}{\gamma^2}),$$

where the second sum represents an absolutely convergent series for all $p \in \mathbb{P}$:

$$\sum_{p\in\mathbb{P},\kappa\ge 2} |a_p(\chi_j, \theta_m, \kappa)p^{-\kappa(s+1-\frac{1}{4k})}| = \sum_{p\in\mathbb{P},\kappa\ge 2} |a_p(\chi_j, \kappa)|p^{-\kappa(Re(s)+1-\frac{1}{4k})} < \infty,$$

since $|a_p(\chi_j, \kappa)| \le \chi_j(1)a_p(1, \kappa) = \frac{\chi_j(1)}{\kappa}$ as remarked on page 9. Therefore

$$\lim_{m\to\infty} \sum_{p\in M_m} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}$$

converges uniformly in $|s| \le r\gamma$ to an analytic function for every χ_j.
For every character $\chi := \chi_j$ we have (Theorem 2.4 and Remark)

$$L_{M_m}(s+1-\frac{1}{4k},\theta_m,\chi) = \frac{\prod_{l=1}^{m_1} L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)}{\prod_{l=1}^{n_1} L_{M_m}(s+1-\frac{1}{4k},\theta_m,\psi_l^*)}.$$

The last statement uses essentially $L_p(s,\theta,\phi_1+\phi_2) = L_p(s,\theta,\phi_1)L_p(s,\theta,\phi_2)$, which is a consequence of the definitions.

$$\begin{aligned} &\log\left(L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)\right) \\ = &\sum_{p\in M_m} \frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_m,\kappa\geq 2} a_p(\varphi_j^*,\theta_m,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}. \end{aligned}$$

Since the series $\sum\limits_{p\in\mathbb{P},\kappa\geq 2} a_p(\varphi_j^*,\theta_m,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}$ is absolutely convergent, we only have to show the convergence of

$$\lim_{m\to\infty}\sum_{p\in M_m}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}.$$

However since φ_l^* is a class function on G and $\chi_1,\ldots,\chi_k$ is a basis of the class functions we get a linear combination $\varphi_l^* = \sum_{j=1}^k r_{j,l}\chi_j$ and therefore

$$\lim_{m\to\infty}\sum_{p\in M_m}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}} = \sum_{j=1}^k r_{j,l}\Big(\lim_{m\to\infty}\sum_{p\in M_m}\frac{\chi_j(\sigma_p)e^{-2\pi i\theta_{m,p}}}{p^{s+1-\frac{1}{4k}}}\Big).$$

converges uniformly in $|s|\leq r\gamma$. This proves that

$\lim\limits_{m\to\infty}\log L_{M_m}(s+1-\dfrac{1}{4k},\theta_m,\varphi_l^*)$ and $\lim\limits_{m\to\infty}\log L_{M_m}(s+1-\dfrac{1}{4k},\theta_m,\psi_l^*)$

converge uniformly on $|s|\leq r\gamma$ to some analytic functions $g_{\varphi_l^*}(\frac{s}{\gamma^2}), g_{\psi_l^*}(\frac{s}{\gamma^2})$. Thus it is clear that the functions $L_{M_m}(s+1-\frac{1}{4k},\theta_m,\varphi_l^*)$ and $L_{M_m}(s+1-\frac{1}{4k},\theta_m,\psi_l^*)$ converge to some holomorphic functions $f_{\varphi_l^*}(\frac{s}{\gamma^2})$ and $f_{\psi_l^*}(\frac{s}{\gamma^2})$ with $f_{\varphi_l^*}\neq 0$ and $f_{\psi_l^*}(s)\neq 0$ on $|s|\leq r\gamma$.
Therefore the conditions in Corollary 3.1 are fulfilled and we find a set A_ϵ such that $\liminf\limits_{T\longrightarrow\infty}\frac{vol(A_\epsilon\cap(0,T))}{T}>0$ and for $|s|\leq r\gamma-\epsilon/2$ and $t\in A_\epsilon$

$$|L(s+1-\frac{1}{4k}+it,\chi_j,K/\mathbb{Q}) - f_j(\frac{s}{r\gamma^2})| < \epsilon/2.$$

If $\epsilon>0$ is chosen sufficiently small, such that $r\leq r\gamma-\epsilon/2$, then for $t\in A_\epsilon$, $|s|\leq r$ and $j=1,\ldots,n$

$$|L(s+1-\frac{1}{4k}+it,\chi_j,K/\mathbb{Q}) - f_j(s)| < \epsilon.$$

□

Remark 5.1. *As in Remark 3.1 we may again replace $\mathbb{P}$ by some set $\mathbb{P} \setminus \{p_1, \dots, p_d\}$ with primes $p_1, \dots, p_d$. Thus the statement of the last Theorem remains true if we replace the Artin L-Series by those series, where the Euler product is just extended over the set $\mathbb{P} \setminus \{p_1, \dots, p_d\}$. These Artin L-Series just differ by a a finite product $\prod_{j=1}^{d} L_{p_j}(s, \chi)$ from the original ones.*
We may even change a finite number of Euler-factors to get the same result.

This follows from Remark 3.1.

CHAPTER 6

Consequences

1. Artin L-Series

We know from Artin [**3**, p.122], that there is no multiplicative relation of the form $\prod_j L(s,\chi_j)^{c_j} = 1$ between the primitive Artin L-series of a normal extension of $\mathbb{Q}$.

THEOREM 6.1. *Suppose that for a continuous function $f : \mathbb{C}^k \longrightarrow \mathbb{C}$ and the primitive Artin L-series $L(s,\chi_j)$ there is a relation of the form*

$$f(L(s,\chi_1),\dots,L(s,\chi_k)) = 0$$

for all $s \in \mathbb{C}$ where these Artin L-series are defined.
Then we have $f \equiv 0$.

<u>Proof</u>: Suppose that $f \not\equiv 0$. Then there is an open set $U \subset \mathbb{C}^k$ such that $f(z) \neq 0$ for all $z \in U$. Because U is open we may find a point $a \in U$ with $a_j \neq 0$ for $j = 1,\dots,k$. According to Theorem 5.1 there is a complex number $s \in \mathbb{C}$ such that $|L(s,\chi_j) - a_j| < \epsilon$ for every $\epsilon > 0$ and all $j = 1,\dots,k$. So we may suppose that the point $b \in \mathbb{C}^k$ with $b_j := L(s,\chi_j)$ is contained in U, and therefore $f(L(s,\chi_1),\dots,L(s,\chi_k)) \neq 0$, contradicting the assumption of the Theorem. □

THEOREM 6.2. *Let $\chi_1,\dots,\chi_n$ be linearly independent characters of the Galois group of a normal extension $K/\mathbb{Q}$. Then the map $\gamma : \mathbb{R} \to \mathbb{C}^{n(m+1)}$ given by*

$$\gamma(t) = \Big(L(\sigma+it,\chi_1), L'(\sigma+it,\chi_1),\dots,L^{(m)}(\sigma+it,\chi_1),\dots,L^{(m)}(\sigma+it,\chi_n)\Big)$$

is everywhere dense in $\mathbb{C}^{n(m+1)}$, if $1 - \frac{1}{2[K:\mathbb{Q}]} < \sigma < 1$. For a given point $a \in \mathbb{C}^{n(m+1)}$ the set of numbers $t \in \mathbb{R}^{>0}$ such that $\|\gamma(t) - a\| < \epsilon$ for some fixed $\epsilon > 0$ is unbounded.

This is a straight forward generalization of Voronin's Theorem [**14**, p.270] on Dirichlet L-functions.

Proof: Let $(a_0(\chi_1), \ldots, a_m(\chi_1), a_0(\chi_2), \ldots, a_m(\chi_2), \ldots, a_m(\chi_n))$ be any point in $\mathbb{C}^{n(m+1)}$. If one of the $a_0(\chi_j)$ is zero, then replace it by some $b_0(\chi_j) \neq 0$ with $|b_0(\chi_j) - a_0(\chi_j)| < \epsilon$ for some small $\epsilon > 0$.
Because of Theorem 2.13 we only need to approximate the polynomials $p_{\chi_j}(s) := a_0(\chi_j) + \frac{a_1(\chi_j)}{1!}s + \ldots + \frac{a_k(\chi_j)}{k!}s^k$ simultaneously by $L(s + \sigma + it, \chi_j)$ for appropriate values $t \in \mathbb{R}$. This is possible because of Theorem 5.1. Take r small enough such that $p_{\chi_j}(s) \neq 0$ on the disc $|s| \leq r$ and such that $1 - \frac{1}{2[K:\mathbb{Q}]} < \sigma + Re(s) < 1$ for all $|s| \leq r$. We know by Theorem 5.1 that the set A_ϵ is unbounded since

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0.$$

□

For general Artin L-functions $L(s, \phi, K/k)$ with normal extension K/k, Galois group $G(K/k)$ and $k \neq \mathbb{Q}$ we do not get a joint "universality" theorem for linearly independent characters like Theorem 5.1. For example let k be a quadratic number field with class number divisible by a prime $p > 2$ and H_k its Hilbert class field. Then the irreducible characters of $G(H_k/k)$ are all of degree one and also Abelian. We may get $L(s, \chi, H_k/k) = L(s, \overline{\chi}, H_k/k)$ [**3**, p.122/123].

Theorem 6.3. *Let K/k be an arbitrary normal extension with Galois group $G(K/k)$ and ϕ an arbitrary character on $G(K/k)$. Let L with $K \subset L$ be a normal extension of $\mathbb{Q}$ and set $\kappa := [L : \mathbb{Q}]$. r with $0 < r < \frac{1}{4\kappa}$ is fixed. Let $f(s)$ be any function, which is holomorphic on $|s| < r$, continuous for $|s| \leq r$ and $f(s) \neq 0$ for $|s| \leq r$. Then we get a set $A_\epsilon \subset \mathbb{R}$ such that*

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$$

and

$$\forall_{t \in A_\epsilon} \forall_{|s| \leq r} : |L(s + 1 - \frac{1}{4\kappa} + it, \phi, K/k) - f(s)| < \epsilon$$

for the Artin L-function $L(s, \phi, K/k)$.

Proof: Since $L/\mathbb{Q}$ is a normal extension, the same holds for L/k. Therefore we may use Theorem 2.4 (2). We get $L(s, \phi, K/k) = L(s, \phi, L/k)$. The group $G(L/k)$ is a subgroup of $G(L/\mathbb{Q})$. Because of Theorem 2.4 (3) we get $L(s, \phi, L/k) = L(s, \phi^*, L/\mathbb{Q})$. Since ϕ^* is a character of $G(L/\mathbb{Q})$, we have $\phi^* = \sum_{j=1}^{n} m_j \phi_j$, where the ϕ_j, $j = 1, \ldots, n$, are the irreducible characters of $G(L/\mathbb{Q})$, $m_j \in \mathbb{Z}^{\geq 0}$ and one $m_j \geq 1$. Let this be m_1, i.e. $m_1 \geq 1$. We may apply Theorem 5.1 to $L(s, m_j\phi_j, L/\mathbb{Q})$ for those m_j with $m_j \neq 0$. Set $f_1(s) := f(s)$ and $f_j(s) := 1$ for $2 \leq j \leq n$. According to Theorem 2.4 (4) we get

$$L(s, \phi, K/k) = \prod_{\substack{j=1 \\ m_j \neq 0}}^{n} L(s, m_j\phi_j, L/\mathbb{Q}).$$

Therefore the last theorem is a consequence of Theorem 5.1 applied to the Artin L-functions $L(s, m_j\phi_j, L/\mathbb{Q})$ with $m_j \neq 0$. □

2. Zeros of Zeta-Functions

Davenport and Heilbronn [**8**] showed that the ζ-function of an ideal class of a complex quadratic number field has infinitely many zeros in the region $Re(s) > 1$, provided that this number field has class number greater than 1. Voronin proved that these ζ-functions have infinitely many zeros in the strip $1/2 < Re(s) < 1$ [**14**, p.283]. We generalize this result to arbitrary partial ζ-functions attached to any class group of an arbitrary number field, provided that this class group has cardinality greater than 1.

Suppose that $G := I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ is a class group of an arbitrary number field k in the sense of class field theory ([**13**, I; p.4] or [**12**, p.63]). $I^{(\mathfrak{f})}$ is the group of fractional ideals of $\mathcal{O}_k$ prime to $\mathfrak{f}$. $H_{\mathfrak{f}} \subset I^{(\mathfrak{f})}$ is a subgroup with conductor $\mathfrak{f}$ containing the ray of principal ideals
$S_{\mathfrak{f}} := \{\alpha\mathcal{O}_k \mid \alpha \in k \text{ and } \alpha \equiv 1 \mod {}^*\mathfrak{f}\}$.
We remember the definition of the zeta-function of an ideal class $\mathcal{A} \in G$ [**12**, p.100]:

$$\zeta(s, \mathcal{A}) := \sum_{\substack{\mathfrak{a} \in \mathcal{A} \\ \mathfrak{a} \subset \mathcal{O}_k}} \frac{1}{N(\mathfrak{a})^s}, \text{ where } Re(s) > 1.$$

This function may be continued to the entire complex plane $\mathbb{C}$ and has a simple pole at $s = 1$. We get for Hecke L-functions with the Abelian character χ of G [**12**, p.87]

$$L(s, \chi) = \sum_{\mathcal{A} \in G} \chi(\mathcal{A})\zeta(s, \mathcal{A}),$$

where $\chi(\mathcal{A}) := \chi(\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{A}$. We may extend the sum defining the Hecke L-function to a sum over all ideals prime to the conductor of χ and, like in the definition of Artin L-series, to all ideals. However since we wish to use the formula $\zeta(s, \mathcal{A}) = \frac{1}{\#G} \sum_{\chi \in G^*} \overline{\chi(\mathfrak{a})} L(s, \chi)$ with

$\mathfrak{a} \in \mathcal{A}$, we must presuppose that the integral ideals in the classes $\mathcal{A}$ are prime to $\mathfrak{f}$.

THEOREM 6.4. *Suppose that k is a number field, $\mathcal{O}_k$ its ring of integers. Let $H_\mathfrak{f}$ be an ideal group with conductor $\mathfrak{f}$, and $I^{(\mathfrak{f})}$ the group of fractional ideals of $\mathcal{O}_k$ prime to $\mathfrak{f}$. If $I^{(\mathfrak{f})}/H_\mathfrak{f}$ contains more than one ideal class, then the partial ζ-function $\zeta(s, \mathcal{A})$ over any of those classes $\mathcal{A} \in I^{(\mathfrak{f})}/H_\mathfrak{f}$ has infinitely many zeros in the strip $1/2 < Re(s) < 1$.*

If $T > 0$ is sufficiently large, then there is a number $c > 0$ such that there are at least cT zeros of $\zeta(s, \mathcal{A})$ in the region with $1/2 < Re(s) < 1$ and $|Im(s)| < T$.

Proof: From class field theory we know that there is a unique Abelian extension L of k with Galois group $G(L/k)$ and a unique isomorphism $I^{(\mathfrak{f})}/H_\mathfrak{f} \longrightarrow G(L/k)$, called the Artin-Isomorphism. Using this isomorphism Artin proved that every Abelian Artin L-series is a Hecke L-series and vice versa [**3**, p.131, p.171].

So we may proceed by proving our theorem on those Artin L-series attached to $G(L/k)$. There is a unique normal extension $K/\mathbb{Q}$ with $L \subset K$. Every irreducible character χ of $G(L/k)$ may be regarded as a character of $G(K/k)$ by applying the restriction map
$\sigma \in G(K/k) \mapsto \sigma_{|L} \in G(L/k)$, i.e. $\sigma \in G(K/k) \mapsto \chi(\sigma_{|L}) \in \mathbb{C}$.
According to Theorem 2.4 (2) we know that $L(s, \chi, K/k) = L(s, \chi, L/k)$. Further $G(K/k) \subset G(K/\mathbb{Q})$ is a subgroup of $G(K/\mathbb{Q})$. Once again because of Theorem 2.4 (3) we find $L(s, \chi, K/k) = L(s, \chi^*, K/\mathbb{Q})$. The group of all different characters of $I^{(\mathfrak{f})}/H_\mathfrak{f}$ are linearly independent, as well as the characters of $G(L/k)$. The same does not necessarily apply to the induced characters χ^* of the group $G(K/\mathbb{Q})$. However we may prove that the dimension of the subspace spanned by these induced characters is larger than 1:

Suppose that $\chi \neq 1$ is an irreducible character of $G(L/k)$, that is an irreducible character of $G(K/k)$ if we apply the restriction map. Because of Theorem 2.2 we know that $\left(\chi^*, 1\right)_{G(K/\mathbb{Q})} = \left(\chi, 1_{|G(K/k)}\right)_{G(K/k)} = \left(\chi, 1\right)_{G(K/k)} = 0$. The last equation is obvious since $\chi \neq 1$ are both irreducible characters of $G(K/k)$ (they both have degree 1). If we denote the irreducible characters of $G(K/\mathbb{Q})$ by $\phi_1 := 1, \phi_2, \dots, \phi_h$, then for every nontrivial character of $G(L/k)$ we have $\chi^* = \sum_{j=2}^{h} m_j \phi_j$ with $m_j \in \mathbb{Z}^{\geq 0}$. For the induced character 1^* of the trivial character

$1 \in G^*(L/k)$ we get $\big(1^*, 1\big)_{G(K/\mathbb{Q})} = \big(1, 1_{|G(K/k)}\big)_{G(K/k)} = \big(1, 1\big)_{G(K/k)} = 1$. Therefore we have $1^* = \phi_1 + \sum_{j=2}^{h} n_j \phi_j$ with $n_j \in \mathbb{Z}^{\geq 0}$.

So we get

$$L(s, 1) = L(s, 1^*, K/\mathbb{Q}) = L(s, \phi_1, K/\mathbb{Q}) \prod_{j=2}^{h} L(s, \phi_j, K/\mathbb{Q})^{n_j}$$

and for the non-trivial Abelian characters

$$L(s, \chi) = \prod_{j=2}^{h} L(s, \phi_j, K/\mathbb{Q})^{m_j}.$$

Since the irreducible characters ϕ_j are linearly independent, we may apply Theorem 5.1 and Remark 5.1 to $L(s, \phi_1, K/\mathbb{Q})$ and $L(s, \phi_j, K/\mathbb{Q})$ with $2 \leq j \leq h$. Set $\kappa := \#G(K/\mathbb{Q})$. We may therefore find for every $\epsilon_1 > 0$ a set A_{ϵ_1} such that for every $t \in A_{\epsilon_1}$ and for fixed $r < \frac{1}{4\kappa}$ we get $|L(s + it + 1 - \frac{1}{4\kappa}, \phi_j, K/\mathbb{Q}) - 1| < \epsilon_1$ for all $2 \leq j \leq h$ and $|L(s + it + 1 - \frac{1}{4\kappa}, \phi_1, K/\mathbb{Q}) - \big(s - \sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})}\big)| < \epsilon_1$, if $|s| \leq r$. We have $s - \sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})} \neq 0$ for $|s| < 1/2$ since $\sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})} = -1$, if $\mathfrak{a}$ is not in the principal class of $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ and $\sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})} \geq 1$ if $\mathfrak{a} \in H_{\mathfrak{f}}$ [**12**, p.86].

This gives $|L(s + it + 1 - \frac{1}{4\kappa}, 1) - \big(s - \sum_{\chi \neq 1} \overline{\chi(\mathfrak{a})}\big)| < \epsilon$ and for $\chi \neq 1$ $|L(s + it + 1 - \frac{1}{4\kappa}, \chi) - 1| < \epsilon$ for all $|s| \leq r$ and all $t \in A_\epsilon$ in some set A_ϵ.

Take some integral ideal $\mathfrak{a}$ from the class $\mathcal{A}$. We have $\zeta(s, \mathcal{A}) = \frac{1}{\#G} \sum_{\chi \in G^*} \overline{\chi(\mathfrak{a})} L(s, \chi)$. Therefore $|\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A}) - \frac{s}{\#G}| < \epsilon$ for all $|s| \leq r$ and all $t \in A_\epsilon$ as a result of the preceeding.

Suppose that $\epsilon < \frac{r}{\#G}$. Then

$$|\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A}) - \frac{s}{\#G}| < |\frac{s}{\#G}|$$

on the circle $|s| = r$. Inside the disc $|s| < r$ there is exactly one zero of the function $s \mapsto s$. According to Theorem 2.10 we obtain the same number of zeros for the function $\zeta(s + it + 1 - \frac{1}{4\kappa}, \mathcal{A})$ in the disc $|s| < r$

and every fixed $t \in A_\epsilon$. Noting that $c := \liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$ we have completed the proof. $\square$

Suppose that a number field k has class number greater than 1. Its signature is r_1, r_2 and its degree $N := [k : \mathbb{Q}]$. Denote its discriminant by D_k and its different by $\mathfrak{D}_k$. Set $Z(s, \mathcal{A}) := \sqrt{\frac{D_k}{4^{r_2}\pi^N}}^{\,s} \Gamma(\frac{s}{2})^{r_1} \Gamma(s)^{r_2} \zeta(s, \mathcal{A})$, where $\mathcal{A}$ is an arbitrary class of the class group of $\mathcal{O}_k$. Denote by $\mathcal{A}'$ the class with the property $\mathcal{A}\mathcal{A}' = \mathfrak{D}_k$ in the class group. The function $Z(s, \mathcal{A})$ has the following well known functional equation: $Z(s, \mathcal{A}) = Z(1 - s, \mathcal{A}')$ [**16**, p.254]. Because of the preceeding theorem this ζ-function has zeros in the strip $\frac{1}{2} < Re(s) < 1$.

3. Dedekind Zeta-Functions and Hecke L-Functions

THEOREM 6.5. *Let $K_1, \dots, K_r$ be finite normal extensions of $\mathbb{Q}$ with $K_i \cap K_j = \mathbb{Q}$ for $i \neq j$.*
If for a continuous function $f(x_1, \dots, x_r)$ the equation

$$\forall_{s \in \mathbb{C} \setminus \{0\}} f(\zeta_{K_1}(s), \dots, \zeta_{K_r}(s)) = 0$$

holds, then

$$f \equiv 0.$$

<u>Proof</u>: We have according to Corollary 2.1

$$\zeta_K(s) = \zeta(s) \prod_{\chi \neq 1} L(s, \chi)^{\chi(1)},$$

where the product is taken over all non-trivial irreducible characters of the Galois group of the normal extension $K/\mathbb{Q}$.
These characters χ and the character $1 = \mathrm{id}_{G(K/\mathbb{Q})}$ are a basis of the class functions on the group $G := G(K/\mathbb{Q})$.

Let K be the smallest field that contains all $K_1, \dots, K_r$. K is a finite normal extension of $\mathbb{Q}$. The corresponding irreducible characters of $G(K_j/\mathbb{Q})$ may be regarded as characters of $G(K/\mathbb{Q})$ by using the restriction maps $\sigma \in G(K/\mathbb{Q}) \mapsto \sigma_{|K_j} \in G(K_j/\mathbb{Q})$. Since $G(K/\mathbb{Q}) \cong \prod_j^r G(K_j/\mathbb{Q})$ is a direct product, they are linearly independent. Let $a \in \mathbb{C}^r$ be any point for which $f(a_1, \dots, a_r) \neq 0$, then there is an open subset $U \subset \mathbb{C}^r$ containing a, on which $f(x_1, \dots, x_r) \neq 0$ for all $x \in U$. Therefore we may suppose that $a_j \neq 0$. According to Theorem 5.1 we find for every $\epsilon > 0$ a value $s \in \mathbb{C}$, such that

$\max_{j=1}^{r} |\zeta_{K_j}(s) - a_j| < \epsilon$, that is $(\zeta_{K_1}(s), \ldots, \zeta_{K_r}(s)) \in U$ for small ϵ. This completes the proof. □

In general we cannot prove that for different Galois extensions K_j of $\mathbb{Q}$ the corresponding Dedekind-ζ-functions are algebraically independent. For example let $K := \mathbb{Q}(\xi)$ be the field, where ξ is a primitive 8th root of unity. This extension has 3 different subextensions K_j of degree 2 over $\mathbb{Q}$. We find the algebraic relation $\zeta_K \zeta_{\mathbb{Q}}^2 = \zeta_{K_1}\zeta_{K_2}\zeta_{K_3}$.
More generally as $\zeta_K(s) = \zeta(s) \prod_{\chi \neq 1} L(s,\chi)^{\chi(1)}$ for every normal field it is clear that if $G_K := G(K/\mathbb{Q})$ has more normal subgroups than conjugacy classes, then there is a non-trivial algebraic relation between the corresponding ζ-functions.
Further algebraic relations are discussed in the article of Richard Brauer [**6**].

THEOREM 6.6. *Suppose that we have finite normal extensions $K_j/\mathbb{Q}$, $j = 1, \ldots, n$ and the corresponding Dedekind Zeta-functions ζ_{K_j} do not satisfy any non-trivial algebraic relation.*
Then for every continuous function $f(x_1, \ldots, x_n)$ on $\mathbb{C}^n$ the relation $f(\zeta_{K_1}, \ldots, \zeta_{K_n}) \equiv 0$ implies $f \equiv 0$.

Proof: To prove this, let K be the minimal subfield of $\mathbb{C}$ containing all $K_1, \ldots, K_n$. This field K is a normal extension of $\mathbb{Q}$. We may regard all the characters as characters of $G := G(K/\mathbb{Q})$ by using the restriction map $\sigma \in G(K/\mathbb{Q}) \mapsto \sigma_{|K_j} \in G(K_j/\mathbb{Q})$. The kernel of this homomorphism is a normal subgroup $N_j \lhd G$. Then for the ζ-functions we have $\zeta_{K_j}(s) = \zeta(s) \prod_{\chi \neq 1} L(s,\chi)^{\chi(1)}$, where the product is taken over all characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$. (Theorem 2.4 (2) and Corollary 2.1.)

According to Theorem 5.1 we can approximate all values $y_1 \neq 0, y_\chi \neq 0$ simultaneously by $\zeta(s)$ and $L(s,\chi)$ for $\chi \neq 1$ by taking a suitable $s \in \mathbb{C} \setminus \{1\}$.
To prove the theorem it has to be shown that the same holds for the $X_{K_j} := y_1 \prod_{\chi \neq 1, K_j} y_\chi$ (the index K_j indicates that the product is taken over the characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$): i.e., every set of non-zero values $X_{K_j}, j = 1, \ldots, n$ can be simultaneously approximated.

Taking the logarithms $\log X_{K_j} = \log y_1 + \sum_{\chi \neq 1, K_j} \log y_\chi$ (each sum is taken over all χ with $\chi(x) = \chi(1)$ for all $x \in N_j$) we get n linear equations in the variables $\log y_1, \log y_\chi$ for $\chi \neq 1$. The variables X_j can be simultaneously approximated if the right sides of these equations are

linearly independent.
However if these equations were not linearly independent, then there would be a relation $0 = \sum_{j=1}^{n} m_j \Big(\log y_1 + \sum_{\chi, K_j} \log y_\chi \Big)$ with integers $m_j \neq 0$ for some j. This would result in an algebraic relation $\prod_{j=1}^{n} \zeta_{K_j}^{m_j}(s) = 1$ between the $\zeta_{K_j}(s)$. □

THEOREM 6.7. *Let K be a number field, ζ_K the corresponding Dedekind-ζ-function. Let $f(x_1, \ldots, x_m)$ be a continuous function, then the differential equation $f(\zeta_K, \zeta_K', \ldots, \zeta_K^{(m)}) \equiv 0$ implies $f \equiv 0$.*

Proof: Denote by H_K the Hilbert class field of K. Denote the principal character on $G(H_K/K)$ by 1. Then $\zeta_K(s) = L(s, 1, H_K/K)$ because of Theorem 2.4. Denote by L the normal extension of H_K over $\mathbb{Q}$. We know $L(s, 1, H_K/K) = L(s, 1, L/K) = L(s, 1^*, L/\mathbb{Q})$ as a consequence of Theorem 2.4. We have $1^* = \sum_{\phi} n_\phi \phi$, where the ϕ's are the irreducible characters of $G(L/\mathbb{Q})$, $n_\phi \in \mathbb{Z}^{\geq 0}$ and at least one $n_\phi \geq 1$. Denote this character by ϕ_0 and set $n_0 := n_{\phi_0}$. We get

$$\zeta_K(s) = \prod_{n_\phi \neq 0} L(s, n_\phi \phi, L/\mathbb{Q}).$$

If $f \not\equiv 0$, then there is an open set U such that $f(a) \neq 0$ for all $a \in U$. We may suppose that $a_0 \neq 0$, since the set is open, and that all points b with $|b_j - a_j| < \epsilon$ are also in U. Set $P(s) := a_0 + a_1 s + \ldots + \frac{a_m}{m!} s^m$. We may suppose that $P(s) \neq 0$ on a disc $|s| \leq r$ for some small r since $a_0 \neq 0$. Set $\sigma := 1 - \frac{1}{4[L:\mathbb{Q}]}$. According to Theorem 5.1 we may find for every $\epsilon_1 > 0$ numbers $t \in \mathbb{R}$, such that $|L(s + \sigma + it, n_\phi \phi, L/\mathbb{Q}) - 1| < \epsilon_1$ and $|L(s + \sigma + it, n_0 \phi_0, L/\mathbb{Q}) - P(s)| < \epsilon_1$. Therefore for every $\epsilon_2 > 0$ we find values $t \in \mathbb{R}$ such that $|\zeta_K(s + \sigma + it) - P(s)| < \epsilon_2$. For $\epsilon_2 > 0$ sufficiently small we thus get $|\zeta_K^{(j)}(\sigma + it) - a_j| < \epsilon$ as a consequence of Theorem 2.13 for $j = 0, \ldots, m$. Therefore $f(\zeta_K(s'), \ldots, \zeta_K^{(m)}(s')) \neq 0$ for this point $s' := \sigma + it$. □

This theorem was already proved by Reich [**26**] by different means. We may prove the analogous statement for arbitrary Hecke L-functions. The only difference in the proof is, that we replace the trivial ray character 1 by an arbitrary ray character χ of a general class group $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ and the Hilbert class field H_K is replaced by the class field attached to this class group $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ [**13**, I,p.9]:

THEOREM 6.8. *Let K be a number field, $I^{(\mathfrak{f})}/H_{\mathfrak{f}}$ a general class group in the sense of class field theory* [**13**] *and χ an Abelian character on this group. Denote by $L(s,\chi)$ the corresponding Hecke L-function. If $f(x_1,\dots,x_m)$ is any continuous function, then the differential equation $f(L(s,\chi),L'(s,\chi),\dots,L^{(m)}(s,\chi))\equiv 0$ implies $f\equiv 0$.*

THEOREM 6.9. *Let $L(s,\chi)$ be a Hecke L-function of a number field K attached to ray class character χ. Suppose that the equation*

$$\sum_{k=0}^{N} s^k F_k(L(s,\chi),L'(s,\chi),\dots,L^{(m)}(s,\chi))=0$$

holds for all $s\in\mathbb{C}$ and fixed continuous functions $F_k:\mathbb{C}^{m+1}\to\mathbb{C}$. Then we get $F_k\equiv 0$ for $k=0,\dots,N$.

Proof: Denote by K_χ the class field attached to the character χ [**7**, p.219]. (This is the class field of the ideal group $H_\chi:=\{\mathfrak{a}\mid\chi(\mathfrak{a})=1\}$ [**12**, p.88].) Then we have $L(s,\chi,K_\chi/K)=L(s,\chi)$. The character χ may be regarded as a character of $G(K_\chi/K)$ by using the Artin-Isomorphism. Denote the normal extension of $\mathbb{Q}$, which contains K_χ, by L. Then because of Theorem 2.4

$$L(s,\chi,K_\chi/K)=L(s,\chi,L/K)=L(s,\chi^*,L/\mathbb{Q}).$$

We have $\chi^*=\sum_\phi n_\phi\phi$, where the ϕ's are the irreducible characters of $G(L/\mathbb{Q})$, $n_\phi\in\mathbb{Z}^{\geq 0}$ and at least one $n_\phi\neq 0$. Let this be n_{ϕ_0} with the character ϕ_0. Set $n_0:=n_{\phi_0}$. Then

$$L(s,\chi)=L(s,\chi^*,L/\mathbb{Q})=\prod_{n_\phi\neq 0}L(s,n_\phi\phi,K/\mathbb{Q}).$$

Suppose that $F_N\not\equiv 0$. We get an open set U such that $|F(a)|>c$ for some positive constant $c>0$ and all $a\in U$. Since this set is open we may even suppose that the first coordinate of points in U satisfy $a_0\neq 0$. Further we may suppose that U is contained in a compact set. Set $P(s):=a_0+a_1s+\ldots+\frac{a_m}{m!}s^m$. We may suppose that $P(s)\neq 0$ for all $|s|\leq r$ for some small $r>0$ since $a_0\neq 0$. Set $\sigma:=1-\frac{1}{4[L:\mathbb{Q}]}$. According to Theorem 5.1 we find for every $\epsilon_1>0$ a set A_{ϵ_1} with $\liminf_{T\to\infty}\frac{vol(A_{\epsilon_1}\cap(0,T))}{T}>0$, such that for all $t\in A_{\epsilon_1}$ we have $|L(s+\sigma+it,n_0\phi_0,L/\mathbb{Q})-P(s)|<\epsilon_1$ and $|L(s+\sigma+it,n_\phi\phi,L/\mathbb{Q})-1|<\epsilon_1$ for all $|s|\leq r$. Then $|L(s+\sigma+it,\chi)-P(s)|<\epsilon_2$ for some set A_{ϵ_1}, if we choose ϵ_1 sufficiently small, and because of Theorem 2.13 $|L^{(j)}(\sigma+it,\chi)-a_j|<\epsilon$ for $j=0,\dots,m$.

Thus we have $|F_N(L(\sigma+it,\chi),\ldots,L^{(m)}(\sigma+it,\chi))| > c$ for small $\epsilon > 0$ and for all $t \in A_{\epsilon_1}$. Then

$$\begin{aligned} c &< |F_N(L(\sigma+it,\chi),\ldots,L^{(m)}(\sigma+it,\chi))| \\ &= |\sum_{k=0}^{N-1}(\sigma+it)^{k-N}F_k(L(\sigma+it,\chi),\ldots,L^{(m)}(\sigma+it,\chi))|. \end{aligned}$$

Since $(L(\sigma+it,\chi),\ldots,L^{(m)}(\sigma+it,\chi)) \in U$ is contained in a compact set, the values of the functions F_k are bounded on the set U. However the set A_{ϵ_1} is unbounded and we get an infinite sequence of values $t_l \in A_{\epsilon_1}$ with $t_l \to \infty$. Taking the limit, we get $0 < c \leq 0$ as a contradiction.

□

Symbols

$\mathbb{Z}$	rational integers
$\mathbb{Z}^{\geq 0}$	rational integers ≥ 0
$x \in \gamma \mod \mathbb{Z}$	*page 15*
$\lvert x - x_0 \mod \mathbb{Z}\rvert < \epsilon$	*page 15*
$\mathbb{Q}$	rational numbers
$\mathbb{P}$	rational primes
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	real numbers > 0
$\mathbb{R}^{\mathbb{P}}$	functions $\theta : \mathbb{P} \to \mathbb{R}$
$\theta \in \mathbb{R}^{\mathbb{P}}$	$(\theta_p)_{p\in\mathbb{P}}$
$[\alpha]$	greatest integer such that $[\alpha] \leq \alpha$
$\{\alpha\}$	$\{\alpha\} := \alpha - [\alpha]$
$\mathbb{C}$	complex numbers
$\#M$	cardinality of a finite set M
Y^c	the complement of a set $Y \subset M$
$M \setminus Y$	$\{x \in M \mid x \notin Y\}$
$(a, b]$	$a, b \in \mathbb{R}$, interval $a < x \leq b$
k, K	algebraic number fields
$\mathcal{O}_k$	ring of integers of the number field k
H_k	Hilbert class field of k
K_χ	ray class field of the ray character χ [**7**, p.219]
$G(K/k)$	Galois group of K/k
$[K : k]$	degree of K relative to k
$Trace(\alpha)$	trace of the algebraic number α
$N(\alpha)$	norm of the algebraic number α
$N(\mathfrak{a})$	norm of the ideal $\mathfrak{a}$
$I^{(\mathfrak{f})}$	group of fractional ideals prime to $\mathfrak{f}$
$S_{\mathfrak{f}}$	ray $\mod \mathfrak{f}$
$H_{\mathfrak{f}}$	ideal group with conductor $\mathfrak{f}$
$(\mathfrak{P}, K/k)$	Frobenius Automorphism
$\sigma_{\mathfrak{p}}$	Frobenius Automorphism
χ^*	induced character of χ
$f_{\mid U}$	map f restricted to U

(ϕ, ψ)	scalar product of the class functions ϕ, ψ
$f = O(g)$	Landau symbol
$Re(z)$	real part of $z \in \mathbb{C}$
$Im(z)$	imaginary part of $z \in \mathbb{C}$
$vol(M)$	Lesbegue-measure of a set
$\mathcal{H}$	Hilbert space
$\langle x, y \rangle$	scalar product of $x, y \in \mathcal{H}$
$\|x\|$	norm of a vector
$\|x\|$	for a scalar product: $\|x\| = \sqrt{\langle x, x \rangle}$
$\|L\|$	$= \sup_{\|x\|=1} \|L(x)\|$ for continuous linear operator L
$L(s, \phi, K/k)$	Artin L-function of the non-Abelian character ϕ
$\zeta_k(s)$	Dedekind Zeta-function of the number field k
$L(s, \chi)$	Hecke L-function for ray characters χ or
$L(s, \chi)$	$= L(s, \chi, K/\mathbb{Q})$ for non-Abelian character χ of $G(K/\mathbb{Q})$
$L_M(s, \chi, \theta)$	finite Euler-product, *Def. 6, p.17*
$GL_k(\mathbb{C})$	general linear group of $k \times k$ matrices
$\det(A)$	determinant of a matrix A
E	unit matrix
$U \lhd G$	U is a normal subgroup of G
$L_2(a, b)$	space of square integrable functions with support in (a, b)
$Res_{s=s_0} f(s)$	residue of the function f at $s = s_0$
$v_p(d)$	p-valuation of d for $p \in \mathbb{P}$

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