On the Value Distribution of Artin-L-Series and the Functional Independence of Dedekind-ζ-Functions.

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We generalize a theorem of Voronin [4] on the joint distribution of non-zero values of Dirichlet L-Functions to Artin-L-Functions. As a consequence we get the differential independence of ζ-functions of normal extensions of $\mathbb{Q}$.

1. MAIN THEOREM

We prove the following statement on Artin-L-series over $\mathbb{Q}$:

THEOREM 1.1. *Let K be a finite Galois-extension of $\mathbb{Q}$ and $\chi_1, \ldots, \chi_n$ linearly independent characters of the group $G := Gal(K/\mathbb{Q})$. Let $k := \#G$ and $f_1(s), \ldots, f_n(s)$ be holomorphic functions on $|s| < r$ and continuous on $|s| \leq r$, where r is a fixed number $0 < r < \frac{1}{4k}$. Further suppose $f_j(s) \neq 0$ on $|s| \leq r$.*

Then for every $\epsilon > 0$ there is a set $A_\epsilon \subset \mathbb{R}$ such that

$$\liminf_{T \longrightarrow \infty} \frac{vol(A_\epsilon \cap (0,T))}{T} > 0$$

and for $j = 1, \ldots, n$

$$\forall_{t\in A_\epsilon}\forall_{|s|\le r} : |L(s+1-\frac{1}{4k}+it,\chi_j)-f_j(s)|<\epsilon,$$

where $L(z,\chi_j)$ denotes the corresponding Artin-L-series.

As a consequence we find conditions for the functional independence of Dedekind-ζ-functions (Theorems 4.6 and 4.7) and the differential independence of Dedekind-ζ-functions (Theorem 4.8).

2. PREPARATION

Denote by $\mathbb{P}$ the set of rational primes.

DEFINITION 2.1. Suppose that

$$F(s)=\prod_{p\in\mathbb{P}} f_p(p^{-s})$$

where $f_p(z)$ is a rational function and the product converges absolutely for $Re(s)>1$.

Then for any finite set $M\subset\mathbb{P}$ of primes and for any $\theta\in\mathbb{R}^{\mathbb{P}}$ we define

$$F_M(s,\theta):=\prod_{p\in M} f_p(p^{-s}e^{-2\pi i\theta_p}).$$

LEMMA 2.1. *Suppose that $F_1(s),\ldots,F_n(s)$ are analytic functions which are represented by products*

$$F_j(s)=\prod_{p\in\mathbb{P}} f_{p,j}(p^{-s})$$

for $Re(s)>1$,where $f_{p,j}(z)=1+\sum_{m=1}^{\infty}a_{p,j}^{(m)}z^m$ are rational functions of z without poles in the disc $|z|<1$. For all $\epsilon>0$ there are constants $c(\epsilon)>0$ with

$$|a_{p,j}^{(m)}| \leq c(\epsilon)p^{m\epsilon}.$$

Further suppose that they have an analytic continuation to the plane $Re(s) > 1 - 1/2k$ *with at most one simple pole at* $s = 1$.

Assume that

$$\frac{1}{T}\int_{-T}^{T} |F_j(\sigma + it)|^2 dt$$

is uniformly bounded for $\sigma \in (\alpha, 1)$ *and* $T \in \mathbb{R}^+$, *if* $\alpha \in (1 - \frac{1}{2k}, 1)$ *is fixed.* *Let* $M_1 \subset M_2 \subset \ldots$ *be finite sets of primes with* $\mathbb{P} = \bigcup_{j=1}^{\infty} M_j$.
Suppose $\lim\limits_{j->\infty} F_{j,M_j}(s, \theta_j) = f_j(s)$ *uniformly in* $|s - (1 - \frac{1}{4k})| \leq r < \frac{1}{4k}$.

Then for any $\epsilon > 0$ *there exists a set* $A_\epsilon \subset \mathbb{R}$ *such that for all* j *and all* $t \in A_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})|\leq r} |F_j(s + it) - f_j(s)| < \epsilon$$

and

$$\liminf_{T\longrightarrow\infty} \frac{vol(A_\epsilon \cap (0, T))}{T} > 0.$$

Corollary 2.1. *Let* $G_m(s) := \prod_{b=1}^{b=N_m} F_{m,b}(s) / \prod_{b=1}^{b=N_m^*} F_{m,b}^*(s)$. *Suppose that the functions* $F_{m,b}(s), F_{m,b}^*(s)$ *satisfy the conditions of Lemma 1.* *Assume that* $\lim\limits_{j->\infty} G_{m,M_j}(s, \theta_j) = f_m(s)$ *and* $\lim\limits_{j->\infty} F_{m,b,M_j}(s, \theta_j,) = f_{m,b}(s)$. *Under the further condition that* $\max\limits_{m,b,s} |f_{m,b}(s)| > 0$ *and* $f_m(s) = \prod_{b=1}^{b=N_m} f_{m,b}(s) / \prod_{b=1}^{b=N_m^*} f_{m,b}^*(s)$ *for* $|s| \leq r$ *we have:*

For any $\epsilon > 0$ *there is a set* $B_\epsilon \subset \mathbb{R}$ *such that for all* m *and all* $t \in B_\epsilon$

$$\max_{|s-(1-\frac{1}{4k})|\leq r} |G_m(s + it) - f_m(s)| < \epsilon$$

and

$$\liminf_{T \longrightarrow \infty} \frac{vol(B_\epsilon \cap (0,T))}{T} > 0.$$

Proof. The proof of the lemma is completely analogous to Voronin's proof in [4, p.256]. ▮

LEMMA 2.2. *Let $0 < r < \frac{1}{4k}$ and $\chi_1, \ldots, \chi_n$ be linearly independent non-abelian characters of $G := Gal(K/\mathbb{Q})$ where K is a finite normal algebraic extention of $\mathbb{Q}$. Let $k := \#G$.*
Suppose that $f_1(s), \ldots, f_n(s)$ are analytic for $|s| < r$ and continuous for $|s| \leq r$ and not zero on the disc $|s| \leq r$. Then for every pair $\epsilon > 0$ and $y \in \mathbb{R}^+$ there exists a finite set of primes M containing all primes smaller than y and $\theta \in \mathbb{R}^{\mathbb{P}}$ such that:

$$\max_{j=1}^{n} \max_{|s| \leq r} |L_M(s + 1 - \frac{1}{4k}, \chi_j, \theta) - f_j(s)| < \epsilon$$

For the proof we need the following theorem on conditionally convergent series

THEOREM 2.2. [4, p.352] *Suppose that a series of vectors $\sum_{n=1}^{\infty} u_n$ in a real Hilbert space $\mathcal{H}$ satisfies $\sum_{n=1}^{\infty} \|u_n\|^2 < \infty$ and for every $e \in \mathcal{H}$ with $e \neq 0$ the series $\sum_{n=1}^{\infty} \langle u_n, e \rangle$ converges conditionally.*
Then for any $v \in \mathcal{H}$ there is a permutation π of $\mathbb{N}$ so that $\sum_{n=1}^{\infty} u_{\pi(n)} = v$ in the norm of $\mathcal{H}$.

and a theorem of Artin:

THEOREM 2.3 (Artin). *[2, p.122] If $\pi(C_j, x)$ is the number of primes in the class C_j smaller than x then: $\pi(x, C_j) = \frac{h_j}{k} \int_2^x \frac{dt}{\log t} + O(xe^{-a \log^{1/2} x})$ where a is some positive constant, $k = \#G$ and $h_j := \#C_j$.*

THEOREM 2.4 (Paley-Wiener). [1, p.166] *Let F be an entire function. Then the following statments are equivalent:*

(1) $\int_{-\infty}^{\infty} |F(x)|^2 dx < \infty$ *and* $\limsup_{z\in\mathbb{C}} |F(z)e^{-(\sigma+\epsilon)|z|}| < \infty$ *for every* $\epsilon > 0$

(2) *there is a function* $f \in L^2(\sigma, -\sigma)$ *such that* $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} f(u)e^{iuz} du$

THEOREM 2.5 (Markov). [1, p.314] *Let P be a polynomial of degree $\leq n$. Then* $\max_{|x|\leq 1} |P'(x)| \leq n^2 \max_{|x|\leq 1} |P(x)|$.

Proof (of Lemma 2.2).

Choose $\gamma > 1$ such that $\gamma^2 r < \frac{1}{4k}$ and

$$\forall_j : \max_{|s|\leq r} |f_j(s) - f_j(s/\gamma^2)| < \epsilon$$

Because $f_j(s) \neq 0$ we can write

$$f_j(s) = \exp(g_j(s)) \text{ for some } g_j(s) \text{ analytic in } |s| < \gamma^2 r.$$

Hence it is sufficient to prove the Lemma for the logarithms of the functions.
For Artin-L-series $L(s, \chi_j)$ the Euler-factors are defined by $1/\det(E_{k_j} - \rho_j(\sigma_p)p^{-s})$, where σ_p is one of the conjugate Frobenius-automorphisms over $p \in \mathbb{P}$ and $\rho_j : G \longrightarrow GL_{k_j}(\mathbb{C})$ is a representation of G with $\chi_j(\sigma) = \mathrm{trace}(\rho_j(\sigma))$.
For the Euler-factors of $L_M(s', \chi_j, \theta)$ we get:

$$-\log L_p(s', \chi_j, \theta) = \frac{\mathrm{trace}(\rho_j(\sigma_p)) \exp(-2\pi i \theta_p)}{p^{s'}} + \sum_{m\geq 2} a_{m,p} p^{-ms'}$$

The first term is $\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}}$. Therefore

$$-\log L_M(s',\chi_j,\theta) = \sum_{p\in M} \frac{\chi_j(\sigma_p)e^{-2\pi i\theta_p)}}{p^{s'}} + \sum_{p\in M}\sum_{m\geq 2} a_{m,p}p^{-ms'}$$

The second term is a uniformly and absolutely convergent series for all primes in $\mathbb{Q}$.
We define a real Hilbert space $\mathcal{H}_n^{(R)}$ of holomorphic functions on the disc $|s| \leq R$ with the scalar product

$$\langle (h_j)_{j=1}^n, (f_j)_{j=1}^n \rangle := Re \int\limits_{|s|\leq R} \sum_{j=1}^n f_j(s)\overline{h_j(s)}d\sigma dt.$$

Set $R := \gamma r$ $(\gamma < 1)$ and $\eta_p(s) := (\frac{\chi_j(\sigma_p)\exp(-2\pi i\theta_p)}{p^{s'}})_{j=1}^n$, where $s' = s + 1 - \frac{1}{4k}$ with $|s| \leq R$.

Denote the different conjugacy classes of the group G by $C_1, \ldots, C_N$. Obviously $n \leq N$ since N is the dimension of the vectorspace of class functions on G.

Denote the different prime classes by $\mathbb{P}_j := \{p \mid \sigma_p \in C_j\}$.
To define θ: In the natural order of the sets $\mathbb{P}_j$ such that $p_{j,1} < p_{j,2} < p_{j,3} \ldots$ set $\theta_{p_{j,l}} := \frac{l}{4}$. Thereby θ_p is defined for all primes.
We will use the above Theorem 2.2 on conditionally convergent series in Hilbert spaces.

We only need to show that the series $\eta_p, p \in \mathbb{P}$ fulfills the conditions of this theorem:

$\sum_{p\in\mathbb{P}} \|\eta_p\|^2 \leq kn \sum_{p\in\mathbb{P}} p^{\frac{1}{2k}-2+2R} < \infty$. (obviously $\frac{1}{2k} - 2 + 2R < -(1+\epsilon_1)$ for some small ϵ_1)
For e we can choose any $\varphi(s) \in \mathcal{H}_n^R$ with $\|\varphi(s)\| := \langle \varphi(s), \varphi(s)\rangle^{1/2} = 1$.
Now we have to show that

$$\sum_{p\in\mathbb{P}} \langle \eta_p, \varphi(s)\rangle$$

is conditionally convergent or equivalently,
that there exist two sets of primes $\mathbb{P}_+$ and $\mathbb{P}_-$ such that $\forall_{p\in\mathbb{P}_+} : \langle\eta_p, \varphi(s)\rangle > 0$, $\sum\limits_{p\in\mathbb{P}_+} \langle\eta_p, \varphi(s)\rangle = \infty$, and $\sum\limits_{p\in\mathbb{P}_-} \langle\eta_p, \varphi(s)\rangle = -\infty$, $\forall_{p\in\mathbb{P}_-} : \langle\eta_p, \varphi(s)\rangle < 0$.
We compute:

$$\begin{aligned}
\langle\eta_p, \varphi\rangle &= Re \int\limits_{|s|\le R} \sum_{j=1}^{n} \eta_{p,j}(s)\overline{\varphi_j(s)} d\sigma dt \\
&= Re \int\limits_{|s|\le R} \sum_{j=1}^{n} \chi_j(\sigma_p) e^{-2\pi i\theta_p} p^{-s'} \overline{\varphi_j(s)} d\sigma dt \\
&= Re\big(e^{-2\pi i\theta_p} \int\limits_{|s|\le R} p^{-(s+1-\frac{1}{4k})} \big(\sum_{j=1}^{n} \chi_j(\sigma_p)\overline{\varphi_j(s)}\big) d\sigma dt\big)
\end{aligned}$$

It follows that

$$\lim_{p->\infty} |\langle\eta_p, \varphi\rangle| = 0.$$

Since the characters χ_j are linearly independent, there is a class C_l in G such that $\varphi_0 := \sum\limits_{j=1}^{n} \chi_j(\sigma_p)\overline{\varphi_j(s)} \not\equiv 0$ for all $\sigma_p \in C_l$.
As the functions φ_j are holomorphic in the disc $|s| \le R$, we have $\varphi_0(s) = \sum\limits_{m=0}^{\infty} \alpha_m s^m$.
For $p \in C_l$ we have

$$\begin{aligned}
\langle\eta_p, \varphi\rangle &= Re\big(e^{-2\pi i\theta_p} \int\limits_{|s|\le R} \exp\big(-\log(p)(s+1-\frac{1}{4k})\big)\overline{\varphi_0}(s) d\sigma dt \\
&= Re\big(e^{-2\pi i\theta_p} \Delta(\log p)\big)
\end{aligned}$$

Here $\Delta(x) := \int\limits_{|s|\le R} \exp\big(-x(s+1-\frac{1}{4k})\big)\overline{\varphi_0}(s) d\sigma dt$.

Therefore

$$\begin{aligned}\Delta(x) &= \exp\big(-x(1-\frac{1}{4k})\big)\int\limits_{|s|\le R}\exp(-xs)\overline{\varphi_0}(s)d\sigma dt\\ &= \pi R^2\exp\big(-x(1-\frac{1}{4k})\big)\sum_{m=0}^{\infty}\frac{(-1)^m\overline{\alpha}_m(xR^2)^m}{(m+1)!}\end{aligned}$$

We have

$$\|\varphi_0\|^2=\int\limits_{|s|\le R}|\varphi_0|^2d\sigma dt=\pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1}.$$

Using the continuous linear form $L((f_j)_{j=1}^n):=\sum_{j=1}^n\overline{\chi_j(C_l)}f_j$ we get

$$\|\varphi_0\|^2=\|L(\varphi)\|^2\le\|L\|^2\|\varphi\|^2=\|L\|^2.$$

This gives:

$$\pi R^2\sum_{m=0}^{\infty}\frac{|\alpha_m|^2R^{2m}}{m+1}=\|\varphi_0\|^2\le\|L\|^2.$$

Setting $\beta_m:=(-1)^mR^m\overline{\alpha}_m/(m+1)$ we get $\sum_{m=0}^{\infty}|\beta_m|^2\le\|L\|^2/(\pi R^2)$, which gives us an upper bound for $|\beta_m|$.
Set

$$F(u):=\sum_{m=0}^{\infty}\frac{\beta_m}{m!}u^m.$$

$F(u)$ is an entire function. For any $\delta>0$ there is a sequence $u_n\longrightarrow\infty$ such that

$$|F(u_n)|>\exp\big(-(1+2\delta)u_n\big).$$

Suppose the contrary. Then for all $u\in\mathbb{R}$, some $A>0$ and some small $\delta>0$

$$|e^{(1+\delta)u}F(u)|<Ae^{-\delta|u|}.$$

Because

$$|e^{(1+\delta)z}F(z)| \le e^{3|z|} \text{ we have for all } z \in \mathbb{C}$$

and as $e^{(1+\delta)z}F(z)$ is entire, by Theorem 2.4 we obtain a function f with support in $(-3,3)$ such that

$$e^{(1+\delta)z}F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)e^{iuz}du.$$

Because $|e^{(1+\delta)u}F(u)| \le Ae^{-\delta|u|}$ for $u \in \mathbb{R}$, we get

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(1+\delta)u}F(u)e^{-izu}du,$$

where the equality holds almost everywhere for $z \in \mathbb{R}$. This integral converges absolutely for $|Im(z)| \le \delta/2$. So it defines an analytic function near the real axis. Therefore the support of f in $\mathbb{R}$ can not be in the interval $(-3,3)$. This contradicts the assumption.
We have $\Delta(x) = \pi R^2 \exp\big(-x(1-\frac{1}{4k})\big)F(xR)$. Set $x_n := u_n/R$. Then $|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)$ for $\delta_0 > 0$ sufficiently small.

As a consequence we find subintervals I_n of $[x_n-1, x_n+1]$ of length greater than $\frac{1}{2x_n^8}$ in which one of the the inequalities

$$\begin{aligned} |Re\Delta(x)| &> \frac{e^{-(1-\delta_0)x}}{200} \text{ or} \\ |Im\Delta(x)| &> \frac{e^{-(1-\delta_0)x}}{200} \end{aligned}$$

holds.
To prove this we approximate Δ by polynomials. Set $N := [x_n] + 1$. Let B be an upper bound of the $|\beta_m|$. This gives $|F(xR)| \le Be^{xR}$. For $x \in [x_n - 1, x_n + 1]$ we have (remember $R < r < 1/4k$)

$$\begin{aligned} |\sum_{m=N^2}^{\infty} \frac{\beta_m}{m!}(xR)^m| &\le B\sum_{m=N^2}^{\infty} \frac{1}{m!}(xR)^m \le B\frac{(xR)^{N^2}}{N^2!}\sum_{m=0}^{\infty}\frac{1}{m!}(xR)^m \\ &\le B\frac{N^{N^2}}{N^2!}e^N \le B\Big(\frac{N}{N^2/e}\Big)^{N^2}e^N \le B\frac{e^{N^2+N}}{N^{N^2}} \le e^{-2x_n} \end{aligned}$$

if x_n is sufficiently large.
Similarly for $x \in [x_n - 1, x_n + 1]$ we have

$$\sum_{N^2=m}^{\infty} \frac{\big(-(1-\frac{1}{4k})x\big)^m}{m!} \leq e^{-2x_n} \text{ and } \exp\big(-(1-\frac{1}{4k})x\big) \leq e^{(1-\frac{1}{4k})x}.$$

Hence $F(xR) = P_1(x) + O(e^{-2x_n})$ and $\exp(-(1-\frac{1}{4k})x) = P_2(x) + O(e^{-2x_n})$, where P_1 and P_2 are polynomials of degree N^2. This gives
$\Delta(x) = P_n(x) + O(e^{-x_n})$ all $N = [x_n] + 1$ and $x \in [x_n - 1, x_n + 1]$, where $P_n(x)$ is a polynomial of degree less than N^4.
Since $|\Delta(x_n)| > \exp\big(-(1-\delta_0)x_n\big)$ we have $\frac{1}{2}e^{-(1-\delta_0)x_n} \leq |P_n(x_n)|$ for large n. Set $a := \max_{|x-x_n|\leq 1} |P_n(x)|$. Then there exists a $\xi \in [x_n - 1, x_n + 1]$ such that $a = |P_n(\xi)|$. There exists a $\kappa \in (\xi, x)$ or $\kappa \in (x, \xi)$ such that $|P_n(\xi) - P_n(x)| = |P_n'(\kappa)(x - \xi)|$. Set $\tau := |\xi - x|/N^8$. Then by theorem 2.5 we have $|P_n(\xi) - P_n(x)| \leq \tau a$. If $\tau \leq 1/2$ then $|1 - \frac{P_n(x)}{P_n(\xi)}| \leq 1/2$, therefore $|P_n(x)| \geq \frac{a}{2} \geq \frac{|P_n(x_n)|}{2} \geq \frac{1}{4}e^{-(1-\delta_0)x_n}$ for all x with $|x - \xi| \leq \frac{1}{2N^8}$. It follows that $|\Delta(x)| \geq \frac{1}{8}e^{-(1-\delta_0)x_n} \geq \frac{1}{8e^2}e^{-(1-\delta_0)x} \geq \frac{1}{100}e^{-(1-\delta_0)x}$ for large n.

For $p_r \in \mathbb{P}_l$, and $p_1 < p_2 < \ldots < p_r < \ldots$ we have $\theta_{p_r} = r/4$ which gives $e^{-2\pi i \theta_{p_r}} = i^r$. Therefore

$$\langle \eta_{p_r}, \varphi \rangle = Re(i^r \Delta(\log(p_r))).$$

One of the inequalities above is satisfied infinitely often. Consider the interval $I_n := [\alpha + \beta, \alpha]$ such that on I_n one of the inequalities
$|Im(\Delta(x))| \geq \frac{1}{200}e^{-(1-\delta_0)x}$ or $|Re(\Delta(x))| \geq \frac{1}{200}e^{-(1-\delta_0)x}$ holds and
$\beta \geq \frac{1}{2x_n^8}$.
By theorem 2.3 the number of primes $p \in \mathbb{P}_l$ for which $\log p \in I_n$ is:

$$\begin{aligned} \pi(e^{\alpha+\beta}, C_j) - \pi(e^\alpha, C_j) &= \frac{h_j}{k} \int_{e^\alpha}^{e^{\alpha+\beta}} \frac{dt}{\log t} + O(e^{\alpha+\beta} e^{-a\alpha^{1/2}}) \\ &\geq \frac{h_j}{k} e^\alpha \left(\frac{e^\beta - 1}{\alpha + \beta} + O(\frac{e^\beta}{e^{a\alpha^{1/2}}}) \right) \end{aligned}$$

Since $\beta > \frac{1}{2x_n^8}$, for x_n sufficiently large we get

$$\pi(e^{\alpha+\beta}, C_j) - \pi(e^\alpha, C_j) \geq \frac{h_j}{k} \frac{e^{x_n}}{x_n^{10}}$$

The number of primes p in the interval I_n with $\exp(-2\pi i\theta_p) = 1$, $\exp(-2\pi i\theta_p) = -1, \exp(-2\pi i\theta_p) = i$, or $\exp(-2\pi i\theta_p) = -i$ is therefore more than $\frac{h_j}{k}\frac{e^{x_n}}{4x_n^{10}}$.

Therefore

$$\sum_{\substack{p\in\mathbb{P}_l\cap I_n \\ \mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p))>0}} \langle \eta_p, \varphi\rangle > c_1 e^{\delta_0 x_n/2}$$

for some positive constant c_1. The same holds for a subset of primes with $\mathrm{Re}(e^{-2\pi i\theta_p}\Delta(\log p)) < 0$, the sum is less than $c_1 e^{\delta_0 x_n/2}$. As $x_n \longrightarrow \infty$ the corresponding series diverge to $+\infty$ and $-\infty$. ∎

THEOREM 2.1. *Assume that a Dirichlet series* $\sum_{n=0}^{\infty} a_n n^{-s}$ *satisfies* $a_n = O_\epsilon(n^\epsilon)$ *for every* $\epsilon > 0$. *Suppose that this series converges for* $Re(s) > 1$ *absolutely and can be analytically continued to the complex plane and has no pole for* $Re(s) \geq 1/2$ *except a simple pole at* $s = 1$. *Denote this function by* $f(s)$. *Suppose further that* $|f(s)|^2 = O(|t|^M)$ *for some* $M := M(a,b) \in \mathbb{R}$ *and* $s = \sigma + it$ *where* $|t| \geq 1$ *and* $\sigma \in [a,b]$ *with* $a, b \in \mathbb{R}$.

Then $\frac{1}{T}\int_{-T}^{T} |f(s+it)|^2\, dt$ *is bounded for every* s *with* $Re(s) > 1 - 1/M$ *and it is uniformly bounded for all* s *with* $Re(s) \geq \alpha$ *where* $1 > \alpha > 1 - 1/M$ *is some fixed number. We can choose* $M = \inf\{m | f(s) = O(|t|^m)\}$.

Proof. Obviously there is a $\xi > 0$ such that

$$\frac{1}{T}\int_{-T}^{T} |f(s)|^2 dt = O(T^\xi)$$

(take for example $\xi := M + 1$).

We denote the infimum of those ξ by μ.

Using a Lemma in [8, p.151], we get for $Re(s) > 1$, $(\delta > 0, c > 1, c > \sigma)$

$$\sum_{n=0}^{\infty} \frac{a_n}{n^s} e^{-\delta n} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw.$$

Because of the condition $a_n = O_\epsilon(n^\epsilon)$ the series on the left side of the equation is absolutely convergent for all $Re(s) > 0$ and therefore it is a holomorphic function in this plane. By Stirling's formula on the Γ-function we get $|\Gamma(s)| \leq C_{[a,b]} |t|^{\sigma-1/2} \exp(-\frac{\pi}{2}|t|)$, where $s = \sigma + it$ and $\sigma \in [a,b]$. Therefore it follows from Cauchy's theorem, that the function $\frac{1}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw$ is an analytic function for all $c > 0$ and $Re(s) > 0$, if $\sigma > \alpha > \sigma - 1$. We have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw =$$

$$\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw + f(s) + \mathrm{Res}_{w=1} \Gamma(w-s) f(w) \delta^{s-w}$$

Set $B := \mathrm{Res}_{s=1} f(s)$. Then we find for f the expression

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} e^{-\delta n} - \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \Gamma(w-s) f(w) \delta^{s-w} dw - B\Gamma(1-s)\delta^{s-1},$$

where $Re(s) \geq 1/2, \sigma > \alpha > \sigma - 1$.

Denote the first term on the right of the last equation by Z_1 and the second term by Z_2.
We have $B\Gamma(1-s)\delta^{s-1} = O(|t|^{1-\sigma-1/2} e^{-\frac{\pi}{2}|t|} \delta^{\sigma-1})$. This implies $B\Gamma(1-s)\delta^{s-1} = O(|t|^{\pi/2} e^{-\frac{\pi}{2}|t|})$ if $\delta \geq |t|^{-\pi/2}, |t| \geq 1$ and $1/2 \geq \sigma \geq 1$.

If $\sigma \geq a > 1/2$ then

$$\begin{aligned}\int_{T/2}^{T} |Z_1|^2 dt &= O(T\sum_{m=1}^{\infty}\frac{|a_m|^2}{m^{-2a}}e^{-2\delta m}) + O\Big(\sum_{m\neq n}\frac{a_m\overline{a}_n e^{-(m+n)\delta}}{m^\sigma n^\sigma |\log(m/n)|}\Big)\\ &= O_a(T) + O(\delta^{2\sigma-2-\epsilon})\end{aligned}$$

for some small $\epsilon > 0$ since $a_n = O(n^\epsilon)$.

Set $w := \alpha + iv$. We obtain

$$\begin{aligned}|Z_2| &\leq \frac{\delta^{\sigma-\alpha}}{2\pi}\int_{-\infty}^{\infty}|\Gamma(w-s)f(s)|dv\\ &\leq \frac{\delta^{\sigma-\alpha}}{2\pi}\Big(\int_{-\infty}^{\infty}|\Gamma(w-s)|dv\int_{-\infty}^{\infty}|\Gamma(w-s)f^2(w)|dv\Big)^{1/2}.\end{aligned}$$

As the first integral is just an integral over the Γ-function, it is bounded. Assume $T \geq |t|$ (recall that $s = \sigma + it$). Set $I_T := (-\infty, -2T] \cup [2T, \infty)$:

$$\int_{I_T}|\Gamma(w-s)f^2(w)|dv = O\Big(\int_{I_T} e^{\frac{\pi}{2}|v-t|}|v|^{-2M}dv\Big) = O\big(e^{-\frac{\pi}{3}T}\big)$$

Hence

$$\begin{aligned}\int_{T/2}^{T}|Z_2|^2 dt &= O\Big(\delta^{2\sigma-2\alpha}\frac{T}{2}O(e^{-\frac{\pi}{3}T}) + \delta^{2\sigma-2\alpha}\int_{-2T}^{2T}|f(w)|^2(\int_{T/2}^{T}|\Gamma(w-s)|dt)dv\Big)\\ &= O(\delta^{2\sigma-2\alpha}) + O\Big(\delta^{2\sigma-2\alpha}\int_{-2T}^{2T}|f(w)|^2dv\Big) = O\big(\delta^{2\sigma-2\alpha}T^{1+2M}\big)\end{aligned}$$

This gives (the bound above is uniform for $\sigma \geq a$):

$$\int_{T/2}^{T} |f(s)|^2 dt = O(T) + O(\delta^{2\sigma-2\alpha-\epsilon}) + O(\delta^{2\sigma-2\alpha}T^{1+\mu+\epsilon}).$$

Set $\delta := T^{-\frac{1}{2}(\frac{1+\mu}{1-\alpha})}$, then for $\sigma > 1 - \frac{1-\alpha}{1+\mu}$ we get

$$\int_{T/2}^{T} |f(s)|^2 dt = O_a(T).$$

Adding up $\int_{T/2}^{T} |f(s)|^2 dt + \int_{T/4}^{T/2} |f(s)|^2 dt + \int_{T/8}^{T/4} |f(s)|^2 dt + \ldots$ gives $\int_{1}^{T} |f(s)|^2 dt = O_a(T)$ and analogously $\int_{-T}^{1} |f(s)|^2 dt = O_a(T)$.

For $\alpha \longrightarrow 0$ and $M \longrightarrow 1+\mu$ we get $Re(s) > 1 - \frac{1}{M}$ as a sufficient condition for $\frac{1}{T}\int_{-T}^{T} |f(s+it)|^2$ being bounded. ∎

REMARK 2.1. For Hecke-L-series over a field k with $\mathbb{Q} \subset k \subset K$, where K is a finite normal extension of $\mathbb{Q}$ the conditions of the Theorem 2.1 are satisfied with $M = [K : \mathbb{Q}]$.

Proof. Denote the Dirichlet-coefficients of the Hecke-L-series $L(s,\chi)$ by $a_n(\chi)$ and the Dirichlet coefficients of ζ_k by a_n. Then we have $|a_n(\chi)| \leq a_n$, where a_n is the number of ideals of norm n in the ring of integers of k. Therefore we have $|a_n| = O_\epsilon(n^\epsilon)$.

Every Hecke-L-series satisfies a functional equation.
$\Lambda(s,\chi) := C^s\Gamma(\frac{s+1}{2})^a\Gamma(\frac{s}{2})^{r_1-a}\Gamma(s)^{r_2}L(s,\chi)$,
where r_1 is the number of real embedings of k, r_2 the number of complex embedings of k, a is the number of infinite places of the conductor of χ and $C \in \mathbb{R}^{>0}$ is a constant. Then $r_1 + 2r_2 = [k : \mathbb{Q}] \leq [K : \mathbb{Q}]$. We have $\Lambda(s,\chi) = W\Lambda(1-s,\chi)$, where W is a root of unity. $L(s,\chi)$ is a holomorphic function for all $s \in \mathbb{C}$, if $L(s,\chi) \neq \zeta_k$. If $L(s,\chi) = \zeta_k$ there is a simple

pole at $s=1$.

By a theorem of Lavrik [6, (p.133: Lemma 2.1)] we have:
$\Lambda(s,\chi)=\frac{c}{s(1-s)}+\sum_{n=1}^{\infty}(a_n f(\frac{C}{n},s)+W\bar{a}_n f(\frac{C}{n},1-s))$, where c is a constant for ζ_k and zero in all other cases.
$f(x,s)=\frac{1}{2\pi i}\int_{\delta-\infty i}^{\delta+\infty i} x^z\Gamma(\frac{z+1}{2})^a\Gamma(\frac{z}{2})^{r_1-a}\Gamma(z)^{r_2}\frac{dz}{z-s}$, where $\delta\in\mathbb{R}$ and $\delta>\max\{\mathrm{Re}(s),0\}$. If we take $\delta>\max\{\mathrm{Re}(s)+1,0\}$, then
$|f(x,s)|\leq\frac{x^\delta}{2\pi}\left(\int_{-\infty}^{\infty}\prod_{k=1}^{r_1}|\Gamma(\frac{\delta+it+s_k}{2})|\right)|\Gamma(\delta+it)|^{r_2}dt=C_\delta x^\delta$
But this means for $Re(s)\in[-1,2]$ that $|\Lambda(s,\chi)|\leq C_\delta 2\sum_{n\in\mathbb{N}}|a_n(\chi)|\frac{1}{n^\delta}$, where $\delta>3$. Therefore $|\Lambda(s,\chi)|\leq 2C_\delta\zeta_k(4)$. The same holds for ζ_k if we suppose that $|Im(s)|$ is big enough, such that we can ignore $\frac{c}{s(1-s)}$. By the well known properties of the Γ-function we get therefore $L(s,\chi)=O(\exp(A|t|)$ and $\zeta_k(s)=O(\exp(A|t|)$ for every fixed strip $Re(s)\in[a,b]$, $Im(s)=t$ and some $A\in\mathbb{R}^{>0}$. To apply the Phragmen-Lindeloef-principle [5], we must show that we have $L(s,\chi)=O(|t|^M)$on the borders $Re(s)=-\epsilon$ and $Re(s)=1+\epsilon$ for large $t=Im(s)$ and every fixed small $\epsilon>0$. This would imply that $L(s,\chi)=O(|t|^M)$ for all $Re(s)\in[-\epsilon,1+\epsilon]$ and $|Im(s)|=|t|>1$.

Then the series $L(s,\chi)$ and $\zeta_k(s)$ converges absolutely for all s with $Re(s)=1+\epsilon$ and we have $|L(s,\chi)|\leq\zeta_k(1+\epsilon)$ and
$|\zeta_k(s)|\leq\zeta_k(1+\epsilon)$. This is an absolute constant independent of $Im(s)=t$. By the functional equation we find that $|L(s,\chi)|=O_\epsilon\big(g(|t|)\big)$ and $|\zeta_k(s)|=O_\epsilon\big(g(|t|)\big)$ for s with $Re(s)=-\epsilon$, where

$$g(|t|)=|\Gamma(\tfrac{s+1}{2})^a\Gamma(\tfrac{s}{2})^{r_1-a}\Gamma(s)^{r_2}/\Gamma(\tfrac{1-s+1}{2})^a\Gamma(\tfrac{1-s}{2})^{r_1-a}\Gamma(1-s)^{r_2}|.$$

Stirling's formula gives $|\Gamma(s)|=O\big(|t|^{\sigma-1/2}\exp(-\frac{\pi}{2}|t|)\big)$, where the constant in the big O depends only on the interval $\sigma\in[a,b]$ with $s=\sigma+it$.
Therefore it follows: $g(|t|)=O\big(|t|^{r_1 2\frac{\sigma}{2}}|t|^{r_2 2\sigma}\big)=O\big(|t|^{\sigma[k:\mathbb{Q}]}\big)$. We had $Re(s)=1+\epsilon$. This means that in the strip $\sigma\in[-\epsilon,1+\epsilon]$ we have $L(s,\chi)=O(|t|^{M_\epsilon})$ and $\zeta_k(s)=O(|t|^{M_\epsilon})$ with $M_\epsilon=(1+\epsilon)[k:\mathbb{Q}]$. Then the infimum

is obviously $[k:\mathbb{Q}]$. ∎

3. PROOF THE MAIN THEOREM

Proof (of theorem 1.1). The theorem of Brauer [7, p.544] states that every character is a finite linear combination $\chi = \sum_l n_l \varphi_l^* - \sum_l m_l \psi_l^*$, where φ_l^*, and ψ_l^* are induced from characters φ_l, ψ_l of degree 1 of subgroups of G. With this theorem Brauer proved, that $L(z,\chi) = \prod_{l=1}^{m_1} L(z,\varphi_l) \Big/ \prod_{l=1}^{n_1} L(z,\psi_l)$, where the series $L(z,\varphi_l)$ and $L(z,\psi_l)$ are Hecke-L-series over number fields contained in K. These are entire functions with the only exception of the Dedekind-ζ-functions which have a simple pole at $z = 1$. Therefore the conditions of Lemma 2.1 are satisfied. Choose the sets $M_\epsilon \subset \mathbb{P}$ according to Lemma 2.2.
Then we have to show that the conditions in Corollary 2 are fulfilled.

If the characters $\chi_1, \ldots, \chi_n$ are not yet a basis of the class functions of G, then add some more characters (for example from the set of irreducible characters of G). Choose additional holomorphic functions f_j, for example constants $\neq 0$, that satisfy the conditions of Lemma 2.2.
As we now have a basis of class functions, every character χ_l^*, ψ_l^* can be expressed as a linear combination of this basis.

We have $|L_{M_\epsilon}(s + 1 - \frac{1}{4k}, \theta, \chi_j) - f_j(s)| < \epsilon$.

Now choose a sequence $\epsilon_n := 1/n$, $y_n := \max M_{n-1}$ $(y_0 := 1)$, $\theta_n \in \mathbb{R}^{\mathbb{P}}$ and $M_n \subset \mathbb{P}$ such that Lemma 2.2 with $\epsilon = 1/n$, $y := y_n$ and $M = M_n$ is satisfied. $M_n \subset M_{n+1}$ is a consequence.

Then because of $\lim_{n->\infty} L_{M_n}(s + 1 - \frac{1}{4k}, \theta_n, \chi_j) = f_j(s)$ and $f_j(s) \neq 0$, we get:

$$\lim_{n->\infty} \left(\sum_{p \in M_n} \frac{\chi_j(\sigma_p) e^{-2\pi i \theta_p}}{p^{s+1-\frac{1}{4k}}} + \sum_{p \in M_n, \kappa \geq 2} a_p(\chi_j, \theta, \kappa) p^{-\kappa(s+1-\frac{1}{4k})} \right) = \log f_j(s),$$

where the second sum represents an absolutely convergent series for $p \in \mathbb{P}$.

For every character $\chi := \chi_j$ we have

$$L_{M_n}(s+1-\frac{1}{4k},\theta,\chi) = \frac{\prod_{l=1}^{m_1} L_{M_n}(s+1-\frac{1}{4k},\theta,\varphi_l^*)}{\prod_{l=1}^{n_1} L_{M_n}(s+1-\frac{1}{4k},\theta,\psi_l^*)},$$

and

$$\log\left(L_{M_n}(s+1-\frac{1}{4k},\theta,\varphi_l^*)\right) =$$

$$\sum_{p\in M_n} \frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_p}}{p^{s+1-\frac{1}{4k}}} + \sum_{p\in M_n,\kappa\geq 2} a_p(\varphi_j^*,\theta,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}.$$

As the series $\sum_{p\in\mathbb{P},\kappa\geq 2} a_p(\varphi_j^*,\theta,\kappa)p^{-\kappa(s+1-\frac{1}{4k})}$ is absolutely convergent, we only need to show the convergence of

$$\lim_{n\to\infty}\sum_{p\in M_n}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_p}}{p^{s+1-\frac{1}{4k}}}.$$

But since φ_l^* is a class function on G and $\chi_1,\dots,\chi_k$ is a basis of the class functions: $\varphi_l^* = \sum_{j=1}^k r_{j,l}\chi_j$,

$$\lim_{n\to\infty}\sum_{p\in M_n}\frac{\varphi_l^*(\sigma_p)e^{-2\pi i\theta_p}}{p^{s+1-\frac{1}{4k}}} = \sum_{j=1}^k r_{j,l}\Big(\lim_{n\to\infty}\sum_{p\in M_n}\frac{\chi_j(\sigma_p)e^{-2\pi i\theta_p}}{p^{s+1-\frac{1}{4k}}}\Big).$$

Since we have now proved that the logarithms of the sequences of functions $L_{M_n}(s+1-\frac{1}{4k},\theta,\varphi_l^*), L_{M_n}(s+1-\frac{1}{4k},\theta,\psi_l^*)$ converge, it is clear that the sequences themselves converge to some holomorphic functions $f_{\varphi_l^*}, f_{\psi_l^*}$ with $f_{\varphi_l^*}\neq 0$ and $f_{\psi_l^*}(s)\neq 0$ on $|s|\leq r$.
Therefore the conditions in Corollary 2 are fulfilled and the theorem is proved. ∎

4. CONSEQUENCES

THEOREM 4.6. *Let $K_1, \ldots, K_r$ be finite normal extensions of $\mathbb{Q}$ with $K_i \cap K_j = \mathbb{Q}$ for $i \neq j$.*
If for a continuous function $f(x_1, \ldots, x_r)$ the following equation

$$\forall_{s \in \mathbb{C} \setminus \{0\}} f(\zeta_{K_1}(s), \ldots, \zeta_{K_r}(s)) = 0$$

holds, then

$$f \equiv 0.$$

Proof. We have [7, p.547]

$$\zeta_K(s) = \zeta(s) \prod_{\chi \neq 1} L(s, \chi)^{\chi(1)},$$

where the product is taken over all non-trivial irreducible characters of the Galois group of the normal extension $K/\mathbb{Q}$.
These characters χ and the character $1 = \mathrm{id}_{G(K/\mathbb{Q})}$ are a basis of the class functions on the group $G := Gal(K/\mathbb{Q})$.

Let K be the smallest field that contains all $K_1, \ldots, K_r$. K is a finite normal extension of $\mathbb{Q}$. The corresponding irreducible characters of $Gal(K_j/\mathbb{Q})$ may be regarded as characters of $Gal(K/\mathbb{Q})$ since
$Gal(K/\mathbb{Q}) = \prod_j^r Gal(K_j/\mathbb{Q})$ is a direct product. Let $a \in \mathbb{C}^r$ be any point for which $f(a_1, \ldots, a_r) \neq 0$, then there is an open subset $U \subset \mathbb{C}^r$ containing a, on which $f(x_1, \ldots, x_r) \neq 0$. Therefore we may suppose that $a_j \neq 0$. By Theorem 1.1 we find for every $\epsilon > 0$ a value $s \in \mathbb{C}$, such that $\max_{j=1}^{r} |\zeta_{K_j}(s) - a_j| < \epsilon$, that is $(\zeta_{K_1}(s), \ldots, \zeta_{K_r}(s)) \in U$ for small ϵ. This completes the proof. ∎

In general we cannot prove that for different Galois extensions K_j of $\mathbb{Q}$ the corresponding Dedekind-ζ-functions are algebraically independent. For example the field $K := \mathbb{Q}(\xi)$, where ξ is a primitive $8-$th root of unity. This extension has 3 different subextensions K_j of degree 2 over $\mathbb{Q}$. Then we find $\zeta_K \zeta_{\mathbb{Q}}^2 = \zeta_{K_1} \zeta_{K_2} \zeta_{K_3}$ as an algebraic relation.

More generally as $\zeta_K(s) = \zeta(s)\prod_{\chi\neq 1} L(s,\chi)^{\chi(1)}$ for every normal field it is clear that if $G_K := Gal(K/\mathbb{Q})$ has more normal subgroups than conjugacy classes, then there is a non-trivial algebraic relation between the corresponding ζ-functions.
Further algebraic relations are discussed in the article of Richard Brauer [3].

THEOREM 4.7. *Suppose that we have finite normal extensions $K_j/\mathbb{Q}$, $j = 1,\ldots,n$ and the corresponding functions ζ_{K_j} do not satisfy any non-trivial algebraic relation.*
Then for every continuous function $f(x_1,\ldots,x_n)$ on $\mathbb{C}^n$ the relation $f(\zeta_{K_1},\ldots,\zeta_{K_n}) \equiv 0$ implies $f \equiv 0$.

Proof. To prove this, let K be the minimal subfield of $\mathbb{C}$ containing all $K_1,\ldots,K_n$. We may regard all the characters as characters of $G := Gal(K/\mathbb{Q})$ as $Gal(K_j/\mathbb{Q}) \cong G/N_j$ for a unique normal subgroup $N_j \lhd G$.
Then for the ζ-functions we have $\zeta_{K_j} = \zeta(s)\prod_{\chi\neq 1} L(s,\chi)^{\chi(1)}$, where the product is taken over all characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$.
By theorem 1.1 we can approximate all values $y_1 \neq 0, y_\chi \neq 0$ simultaneously by $\zeta(s)$ and the $L(s,\chi)$ by taking a suitable $s \in \mathbb{C}\setminus\{1\}$.
To prove the theorem it has to be shown that the same holds for the $X_{K_j} := y_1\prod_{\chi\neq 1} y_\chi$ (the product is taken only over all characters χ with $\chi(x) = \chi(1)$ for all $x \in N_j$): i.e., every set of non-zero values $X_{K_j}, j = 1,\ldots,n$ can be simultaneously approximated.
Taking the logarithms $\log X_{K_j} = \log y_1 + \sum_{\chi\neq 1}\log y_\chi$ (each sum is taken over all χ with $\chi(x) = \chi(1)$ for all $x \in N_j$) the statement is clear if the right sides of these equations are linearly independent.
But if these equations were not linearly independent, then there would be a relation $0 = \sum\limits_{j=1}^{n} m_j\left(\log y_1 + \sum_{\chi,K_j}\log y_\chi\right)$ with integers $m_j \neq 0$ for all j.

This would result in an algebraic relation $\prod_{j=1}^{n}\zeta_{K_j}^{m_j}(s) = 1$ between the ζ_{K_j}. ∎

THEOREM 4.8. *Let $K/\mathbb{Q}$ be a normal extension, ζ_K the corresponding Dedekind-ζ-function. If $f(x_1,\ldots,x_m)$ is any continuous function, then the differential equation $f(\zeta_K,\zeta_K',\ldots,\zeta_K^{(m)}) \equiv 0$ implies $f \equiv 0$.*

Proof. By $\zeta_K = \zeta(s)\prod_{\chi\neq 1} L(s,\chi)^{\chi(1)}$ we may reduce the proof to the proof of the following statement:
For every continuous function g the equation
$g(\zeta(s),\ldots,\zeta^{(k)}(s),L(s,\chi),\ldots,L^{(k_\chi)}(s,\chi),\ldots)\equiv 0$ implies that $g\equiv 0$.
Suppose that $g(a_0,\ldots,a_k,a_0\chi,\ldots,a_{k_\chi}(\chi),\ldots)\neq 0$. Because g is continuous we may assume that $a_0\neq 0, a_0\chi\neq 0,\ldots$ and that on an open set containing $(a_0,\ldots,a_k,a_0\chi,\ldots,a_{k_\chi}(\chi),\ldots)$ the function g is non-zero.
By the continuous dependence of the derivatives of an analytic function on compact domains, which is implied by Cauchy's formula, we only need to approximate the polynomial $a_0+\frac{a_1}{1!}s+\ldots\frac{a_k}{k!}s^k$ by $\zeta(s+1-\frac{1}{4k}+it)$ and the polynomials $a_0(\chi)+\frac{a_1(\chi)}{1!}s+\ldots\frac{a_k(\chi)}{k!}s^{k_\chi}$ by the functions $L(s+1-\frac{1}{4k}+it,\chi)$ with suitably chosen $t\in\mathbb{R}$ according to theorem 1.1. If a_0 and the $a_0(\chi)$ are nonzero we can always suppose (by choosing $r<\frac{1}{4k}$ sufficiently small) that these polynomials are non-zero in the disc $|s|\leq r$.
As the function g is nonzero in an open set containing the point $(a_0,\ldots,a_0(\chi),\ldots)$ this contradicts

$$g(\zeta(s),\ldots,\zeta^{(k)}(s),L(s,\chi),\ldots,L^{(k_\chi)}(s,\chi),\ldots)\equiv 0. \quad \blacksquare$$

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